Minority Turnout and Representation under Cumulative Voting.
An Experiment.*

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Abstract
Under majoritarian election systems, securing participation and representation of minorities remains an open problem, made salient in the US by its history of voter suppression. One remedy recommended by the courts is Cumulative Voting (CV): each voter has as many votes as open positions and can cumulate votes on as few candidates as desired. Theory predicts that CV encourages the minority to overcome obstacles to voting: although each voter is treated equally, CV increases minority’s turnout relative to the majority, and the minority’s share of seats won. A lab experiment based on a costly voting design strongly supports both predictions.

JEL codes: C92, D7, D72, K16

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“And so the world watches America—the only great power in history made up of people from every corner of the planet, comprising every race and faith and cultural practice—to see if our experiment in democracy can work. [...] The jury is still out.” (Obama, 2020, p. xvi).

1 Introduction

The fragility of American democracy, rooted historically in slavery, manifests itself in persistent efforts to disenfranchise racial and linguistic minorities, Black Americans first and foremost. Almost 60 years after the Voting Rights Act (VRA), the disputes we continue to witness are reminders of the heightened importance of voters’ participation. In 2012, the Pew Research Center concluded: “The Growing Electoral Clout of Blacks Is Driven by Turnout”.\(^1\) The date is not coincidental: 2012 was the election year for President Obama’s second term, and for the first time Black turnout was higher than White turnout.\(^2\)

Guaranteeing high electoral participation by minorities requires rules about fair and equal access to voting.\(^3\) But that is not enough: as the surge in Black political engagement during the Obama years shows, it also requires giving minorities the realistic chance of a desired outcome. America’s majoritarian electoral system makes this difficult. Without resorting to proportional representation, the courts have mandated modifications to electoral rules in jurisdictions where majoritarian systems effectively disenfranchise the minority. We study one of such remedies: the use of Cumulative Voting in multi-member districts elections.

As noted already by Charles Dodgson (aka Lewis Carroll) in 1884, the core idea is to vary the number of votes that voters can cast for each candidate. Under CV, each voter has as many votes as there are open seats, and the candidates with more votes win, as under simple plurality. However, each voter is allowed to distribute the votes freely among any number of candidates. CV treats every voter equally; yet, a cohesive minority can ensure itself some victories by cumulating its vote.

There are several reasons why facilitating representation via the voting rule

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\(^1\)Taylor (2013).

\(^2\)According to the Census, Black non-Hispanic turnout increased from from 60% to 65% and then to 67% from 2004 to 2008 to 2012.

\(^3\)For a brief panoramic summary of voter suppression in the US and the role of the VRA, see Grofman et al., 1992.
is superior to the design of favorable district borders. Strategic redistricting implies disputes, arbitrariness, and has debated legality. CV applies to elections of multiple representatives and thus requires multi-member districts, greatly reducing the power of gerrymandering. In addition, CV is a flexible remedy, not conditional on unchanging patterns of geographical segregation and not in need of reform as social and political conditions change. Thus it is also protected from the type of litigation that has weakened the VRA.

CV was used for more than 100 years, from 1870 to 1980, to elect representatives to the Illinois State House and is the rule now in tens of local jurisdictions. Outside local politics, it is used to elect corporate boards in approximately 10% of S&P 500 companies.\textsuperscript{4}

Empirically, CV correlates with an increase in the number of elected minority representatives (Brockington et al., 1998, Bowler et al., 2003), and in the public goods provided to minority communities (Pildes and Donoghue, 1995). In addition, its use appears to increase minority participation in the political system (Bowler et al., 2001). Evaluating these empirical results, however, is complicated by the non-random adoption of CV: CV typically follows voting rights litigation, indicating heightened sensitivity to minority representation and stronger minority involvement. The historical evidence then must be accompanied by experimental testing. It is such testing that we conduct in this paper.

Higher observed minority turnout under CV is usually attributed to more optimistic prospects of affecting the electoral outcome. In the lab, the hypothesis maps directly into an experiment where payoffs depend on one’s own group achieving electoral success but voting is individually costly. Do participants, particularly participants on the minority side, overcome those costs more often when votes can be cumulated?

The costly voting model we implement is the classic tool for studying instrumental voting (Riker and Ordeshook 1968, Palfrey and Rosenthal 1983, Ledyard 1984). The model does not explain the level of turnout observed in large elections, but captures well the comparative statics properties of different elections: turnout is predicted to increase when elections are closer, when they are more salient, when voting costs are lower, when the electorate is smaller. It is this type of comparative effect that interests us: does minority turnout increase

\textsuperscript{4}See Bowler et al. 2003 for a short history of CV. Other useful sources are Bowler et al. 1999, Engstrom 2010, Pildes and Donoghue 1995. For a strong defense of CV, see Guinier 1994. Updated information on the current use of CV is reported in fairvote.org.
when the voting rule changes to CV?

We run different experimental treatments, comparing standard one-vote-per-open-seat voting and CV, and varying both the number of seats and the relative size of the minority. For all our parametrizations, theory suggests that CV should increase the minority’s turnout relative to the majority’s, as well as the fraction of seats won by minority candidates. Both predictions are satisfied in every case. The experiment confirms CV’s potential to increase both the minority’s turnout and its electoral success.

To our knowledge, there is no existing theoretical or experimental study of turnout under CV. Previous laboratory experiments on CV (Gerber et al., 1998 and Cooper and Zillante, 2012) focus on the coordination problem the voting rule poses and neglect the impact of the voting rule on voters’ participation decision. We take the opposite approach. The main assumption underlying our analysis is that the coordination problem is addressed by the parties’ leadership, and addressed primarily through the leaders’ choice of the number of party candidates.

We make this assumption because it mirrors our reading of CV’s historical experiences. For example, Bowler et al. (1999) is a very lively study of CV in Victorian England, an interesting environment for its experimental spirit, the richness of cases, and the availability of historical documents. The authors ask exactly how the strategic problems posed by CV were managed. They find: “a willing demand for party organization from voters, as much as a willing supply of it from the parties themselves”. Strategic mistakes were made, typically in the form of over-nominations by the majority party, but their responsibility was attributed to party leaders, and quickly corrected. Consider the following extracts from Parliamentary hearings on CV (cited by Bowler et al., p. 991):

Mr. Courtney: If a party ran too many candidates it might not gain its due proportion of power.

Mr. Sanford: Quite so. That is its own fault.\footnote{Report and Minutes of Evidence of the Select Committee on School Board Voting, P.P. (1884/ p. 78).}

Or, in an MP’s recollection of a costly instance of over-nominations by the majority party:

Mr. Foster: I suggested whether they could not get the assistance of a well taught child at elementary school who passed a good examination in arithmetic to
show that it was impossible to get the return which they supposed they would.\textsuperscript{6}

Similar sentiments recur in other episodes–see for example Pildes and Donoghue (1995)’s detailed chronicle of the first adoption of CV in Chilton County, Alabama, following VRA litigation. Because they are so costly, nomination mistakes are quickly corrected, and granting party leaders their optimal choice of candidates seems a good working assumption.\textsuperscript{7} Note also that a common finding in the literature is that nomination mistakes are more common on the majority side, for whom the need to concentrate votes is less obvious. If our analysis underestimates the parties’ difficulties in coordinating votes, it is likely to also underestimate the extent to which the minority benefits from CV.

With our focus fully on the turnout decision, we are closer to classic costly voting experiments (Levine and Palfrey, 2007). Because CV is an example of “semi-proportional” voting rules–rules whose results approach proportional representation without imposing proportionality–particularly relevant are the experiments in Herrera et al. (2014) and Kartal (2015), comparing turnout under single-winner majoritarian and proportional elections. However, although CV leads to quasi-proportional outcomes, the turnout decision is quite different: under proportional representation, the value of a marginal single vote is proportional to the change in the party’s vote share, and pivotality, in its usual sense, is moot.\textsuperscript{8} With CV, instead, pivotality continues to drive turnout decisions. The difference, relative to majoritarian voting, is that the possibility of cumulating votes implies richer pivotality calculations. This said, the results are similar: both Herrera et al. and Kartal find that proportional representation increases the turnout rate of the minority relative to the majority’s, and the minority’s expected share of power. The same results hold under CV.

The question then remains of why not to adopt directly a system of proportional representation. The American electoral system has been fashioned by its British roots and by the founders’ resolution to hamper the emergence of factions. Both have resulted in opposition to proportional representation, and its adoption remains highly unlikely. Electoral rules like CV are more familiar to the Anglo-Saxon political (and corporate) environment, have been in use

\textsuperscript{6}Report and Minutes, p. 424. The reference is to the 1870 school board election in Birmingham where the Liberal Party lost its majority. School board elections were important and fiercely fought because they decided religious instruction.

\textsuperscript{7}When CV becomes established, if anything the concern is the possibility of collusion between party leaders, reducing voters’ choices, as was remarked for example during the long experience with CV in the Illinois State House elections (Sawyer and MacRae, 1962).

\textsuperscript{8}Indeed Herrera et al. comment on the similarity in turnout decisions between proportional voting models and non-instrumental models of voting.
since the mid 19th century, and have been studied and at times imposed by the courts. When used, such rules have delivered representation to groups confined to permanent minority status by majoritarian voting and have been able to do so without the rigidity and arbitrariness of majority-minority single member districts, or of quota systems. They deserve more study.

2 Base Model

An electorate of $N$ potential voters selects $K > 1$ representatives for a commission. Each position is identical to the others, and all positions are simultaneously decided in the election. The $N$ voters are divided into two parties: $M$, the majority party with $M$ members, and $m$, the minority party with $m < M$ members, where $M + m = N$. Parties are led by party leaders whose role is to propose the party’s list of candidates.

Party leaders and voters share the same objective: to maximize the number of positions won by their party. The utility derived from one’s party winning $k$ positions is $u(k)$, increasing in $k$. We denote by $V$ the value of controlling all positions and assume $u(k) = (k/K)V$. Linearity captures the focus on the number of positions and simplifies both the lab implementation and the theoretical analysis. We adopt it on substantive grounds as well: any “place at the table” has value. The assumption mirrors an exercise of committee power that is proportional to the number of seats a party has won.

Each voter has $K$ votes, and the $K$ candidates with most votes are elected. If there are ties, after the highest voted candidates are elected, the remaining open positions are filled by selecting winners randomly among the tied candidates. We call $x_p$ the profiles of votes cast by members of party $p$, where $x_{ip}^k$ is the number of votes cast by voter $i \in p$ for candidate $k$, and $x_{ip}$ the vector of all votes cast by $i$.

We study two electoral systems, multi-seat plurality (MP), and cumulative voting (CV). Under MP, each voter casts at most one vote for each candidate: $x_{ip}^k \in \{0, 1\}$ for all $i, k$, and $p$, and each party nominates $K$ candidates. Under CV, each voter can distribute the $K$ votes in any manner the voter desires, as long as the overall budget of $K$ votes is satisfied: $\sum_k x_{ip}^k \leq K$. The possibility of

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9Similarly to CV, Limited Vote (LV) also results in semi-proportional outcomes. Under LV, voters have fewer votes than the number of candidates and cast one vote per chosen candidate. LV is considered simpler than CV but less reliable in generating minority representation. See for example Arrington and Ingalls (1998). For a broad discussion of alternative rules and proportional representation, see Lijphart and Grofman (1984).
cumulating votes creates a coordination problem that party leaders help address by selecting the number of candidates, $G$ for the majority party, and $g$ for the minority party. In line with historical experience\textsuperscript{10}, we allow for fractional votes, but voters, candidates, and positions are constrained to be integers.

The game has two stages. In the first stage, party leaders announce the party list; in the second stage, voters distribute their votes over the party candidates. We focus on equilibria in weakly undominated strategies where voters cast all their votes and cast votes on party candidates only. Under MP, each voter casts one vote for each party candidate. Under CV, the equilibrium is a pair of vote profiles $\{x_M(G,g), x_m(G,g)\}$ and a pair of party lists $\{G(x_M,x_m), g(x_M,x_m)\}$ such that each party member’s votes maximize the number of seats won by the party, given the parties’ lists and the other voters’ voting choices, and each party list maximizes the number of seats won by the party, given the opposite party’s list and all voters’ voting choices.

2.1 Minority representation without voting costs

With no reasons to abstain and no leeway in distributing votes, under MP, party $M$ wins all seats: each $M$ candidate receives $M$ votes, and each $m$ candidate receives $m < M$ votes.

CV grants the minority the possibility of winning some seats. Suppose for example that all voters in party $m$ concentrate all their votes on a single candidate, who thus receives $mK$ votes. The minority wins the seat if its candidate beats the weakest of the majority candidates. If the majority targets all positions, the weakest majority candidate will have most votes when the $MK$ total majority votes are distributed equally among $K$ majority candidates, and each receives $MK/K = M$ votes. Hence the minority can guarantee itself a seat if $mK > M$, or $m > M/K$. This ratio, known in the literature as the threshold of exclusion, is a fraction of $M$: for example, a minority that is half the size of the majority can guarantee itself a seat if the number of open seats is three or more.

Academics and lawyers have extended this logic to a handy formula that delivers a party’s guaranteed number of seats for each $\{m, M, K\}$\textsuperscript{11}. On the

\textsuperscript{10}For example, half votes were allowed in the Illinois State House; half, third, and quarter votes are allowed in the Peoria, IL elections.

\textsuperscript{11}CV-calculators can be found online. See, for example, https://www.lawjock.com/tools/cumulative-voting-calculator/, or Wikipedia https://en.wikipedia.org/wiki/Cumulative_voting. Early influential references in political science are Cole (1950), Glasser (1959), Sawyer and MacRae (1962), Brams (1975), and
face of it, the formula does not address what we are really interested in: not how many seats is the minority sure to win, but how many seats will it win when both parties play their optimal strategies? Yet the answer the formula yields can be grounded in a strategic analysis. In line with our focus on the coordinating role of the party leaders, we call party-optimal those equilibria that for each party maximize the number of seats won, given the votes cast by the opposite party’s voters. We denote by $z$ the number of seats won by party $m$. As we prove in the appendix:

**Proposition 1.** In the absence of voting costs, in all party-optimal equilibria of the CV voting game: (i) for all $m < M/K$, the minority never wins any seat; (ii) for all $m \geq M/K$, $M > m$, and $K$:

$$z \in \left( \frac{Km - M}{M + m}, \frac{Km + M}{M + m} \right)$$

if $\frac{Km - M}{M + m} \notin \mathbb{Z}$

$$z = \begin{cases} 
\frac{Km - M}{M + m} & \text{with prob } m/(m + M) \\
\frac{Km + M}{M + m} & \text{with prob } M/(m + M)
\end{cases}$$

if $\frac{Km - M}{M + m} \in \mathbb{Z}$

Because $(Km + m)/(M + m) - (Km - M)/(M + m) = 1$, if $\frac{Km - M}{M + m}$ is not an integer, there exists a unique integer value of $z$ in the relevant interval; if $\frac{Km - M}{M + m}$ is an integer, then $\frac{Km + M}{M + m}$ is an integer as well.

Suppose, for example, $m = M/2$. Then $z = 1$ if $K = 4$; $z = 2$ if $K = 6$, and $z = 1$ with probability $1/3$ or $2$ with probability $2/3$ if $K = 5$.

The result follows from two main reasons. First, because voters and leaders share a common goal, party-optimal equilibria correspond to the equilibria of the two-player game where the two party leaders directly control the distributions of the votes over the party candidates. Second, the linearity of $u(k)$ renders the game constant-sum. As a result, although there are multiple party-optimal equilibria, all must result in the same payoffs the two parties can guarantee themselves, i.e., the maximin payoffs. Extending the reasoning described above then yields the proposition.

Two observations are useful. First, as an immediate corollary of Proposition 1, there always exists an equilibrium where $g = \lfloor \frac{Km + M}{M + m} \rfloor$, $G = \lfloor \frac{Km + M}{M + m} \rfloor$, and $x^k_{im} = K/g$, $x^k_{1M} = K/G$: all voters spread their votes equally over their party’s candidates, and the two parties nominate just enough candidates to fill all open positions if $\frac{Km - M}{M + m} \notin \mathbb{Z}$, or exceed the number of positions by

1 if $\frac{KM - M}{M + m} \in \mathbb{Z}$. Second, in general many other party-optimal equilibria exist, with different numbers of candidates and/or unequal divisions of votes. For example, suppose $K = 4$, $m = 3$, $M = 6$, and describe an equilibrium by a vector $\{g, G, \{x^n_m\}, \{x^n_M\}\}$. Then $\{1, 3, \{12\}, \{8, 8, 8\}\}$ corresponds to the equilibrium above. But $\{1, 4, \{12\}, \{6, 6, 6, 6\}\}$, or $\{1, 3, \{12\}, \{10, 7, 7\}\}$, or $\{1, 3, \{12\}, \{9, 8, 7\}\}$, or $\{2, 3, \{8, 4\}, \{8, 8, 8\}\}$, or $\{2, 3, \{8, 4\}, \{10, 7, 7\}\}$, or $\{1, 3, \{8, 4\}, \{9, 8, 7\}\}$, all are party-optimal equilibria, and there are many others. The contribution of the proposition is in showing that the multiplicity is irrelevant to the outcome: all party-optimal equilibria must yield the same number of minority victories: $z = 1$ in this example.\(^{12}\)

The strategic problem posed by CV is not trivial.\(^{13}\) Proposition 1 establishes that if party leaders fulfill their coordinating role successfully, the strategic complexity of CV and the large multiplicity of equilibria it can support do not undermine the effect of the voting rule. In contrast to the monopoly of power granted to the majority under MP, CV makes it possible for the minority to win some seats.

This said, in realistic applications a crucial question is voter turnout. We thus must extend the model to include voting costs.

### 3 Voting costs

Suppose now that each voter $i$ faces a cost of voting $c_i$, drawn randomly and independently across voters from a common distribution $F(c)$ with support $[c_\ell, c_\bar{r}]$. Realized costs are private information, but the distribution $F(c)$ is common knowledge and does not depend on party affiliation. The cost $c_i$ represents the cost of going to the polls and is independent of the number of votes cast. A

\(^{12}\)Other equilibria exist that are not party-optimal, where the lack of coordination by the voters of one of the parties prevents it from winning all seats it could win. In the example above, $\{2, 3, \{6, 6\}, \{12, 12, 0\}\}$ is an equilibrium: majority voters fail to coordinate and because each only holds 4 votes, no profitable individual deviation exists. Each party wins two seats.

\(^{13}\)To clarify further the logic of the CV game, it may be useful to differentiate it from a Colonel Blotto game, adapted to the parameters used here. In the Blotto game, two players, with $Km$ and $KM$ tokens respectively, simultaneously distribute them over $K$ boxes; each player earns one point for each box in which the player’s tokens are more numerous than the opponent’s. In the CV game, each of the two players, again endowed with $Km$ and $KM$ tokens respectively, has a separate set of $K$ boxes over which to distribute the tokens; the $K$ boxes with most tokens are chosen, out of the total $2K$ boxes, and each player earns 1 point for each box chosen out of the player’s own set of $K$. The two games are different. For example, in the Blotto game, the equilibrium typically requires mixed strategies, and the player with fewer tokens cannot be guaranteed any points; neither statement applies to the CV game.
voter whose party wins \( k \) positions has utility \( U_i(k) \), given by:

\[
U_i(k) = \begin{cases} 
  u(k) - c_i & \text{if voter } i \text{ voted} \\
  u(k) & \text{if voter } i \text{ abstained}
\end{cases}
\]

### 3.1 Multi-winner plurality (MP)

Under MP, voters who have turned out cast a single vote for each of the party’s \( K \) candidates. Although multiple positions are in play, the analysis mirrors closely the standard approach to costly voting in single winner elections.\(^{14}\)

Call \( S_p \) the number of voters who turn out for party \( p \). Each \( M \) candidate receives \( S_M \) votes, and each \( m \) candidate receives \( S_m \) votes. Thus only three outcomes are possible: either \( S_M > S_m \), and all \( K \) positions are won by \( M \) candidates; or \( S_M < S_m \), and all \( K \) positions are won by \( m \) candidates; or \( S_M = S_m \), and all \( K \) positions are tied, with \( K \) majority and \( K \) minority candidates all having the same number of votes. Under a tie, the \( K \) winners are chosen randomly among all tied candidates. We denote by \( Eu_{\text{MP}}^T \) the expected utility gain from winning seats under MP in case of a tie. Then:

\[
Eu_{\text{MP}}^T = \sum_{k=0}^{K} \binom{K}{k} \left( \frac{K}{2K} \right)^{K-k} u(k) = V/2
\]

where the second equality follows from \( u(k) = (k/K)V \).

As in single-winner elections, a voter from party \( p \) facing opposite party \( p' \) must weigh her cost of voting against the expected utility gain from influencing the outcome. Denoting by \( S_{-ip} \) the number of voters who turn out in party \( p \) ignoring \( i \), voter \( i \) can influence the outcome either by breaking ties (when \( S_{-ip} = S_{p'} \); an event with probability denoted by \( \pi_{p'}^T \)) or by making ties (when \( S_{-ip} = S_{p'} - 1 \), with probability \( \pi_{p'}^{T-1} \)). Thus the thresholds \( \{c_M, c_m\} \) solve the system of equations:

\[
\begin{align*}
    c_m &= \left[ u(K) - Eu_{\text{MP}}^T \right] \pi_m^T (c_M, c_m) + \left[ Eu_{\text{MP}}^T - u(0) \right] \pi_m^{T-1} (c_M, c_m) \\
    c_M &= \left[ u(K) - Eu_{\text{MP}}^T \right] \pi_M^T (c_M, c_m) + \left[ Eu_{\text{MP}}^T - u(0) \right] \pi_M^{T-1} (c_M, c_m)
\end{align*}
\]

\(^{14}\)Arzumanyan and Polborn (2017) study costly voting with multiple candidates but a single winner. Our model is closer to the traditional two-candidate, one-winner set-up, with each party list being the parallel to the party candidate.
or:

\[
\begin{align*}
    c_m &= \frac{V}{2} \pi_m (c_M, c_m) \\
    c_M &= \frac{V}{2} \pi_M (c_M, c_m)
\end{align*}
\]

where \( \pi_p = \pi_p^T + \pi_p^{T-1} \) is the pivotal probability for a voter of party \( p \).

The linearity of the utility function implies that the equilibrium equations (3) and (4) do not depend on \( K \). The problem is then formally identical to the classic costly voting problem with a single winner and two alternatives. It is well-known, and we leave the expressions for the pivot probabilities to the appendix. Given equilibrium \( \{c_m, c_M\} \), we can derive the probabilities of winning different number of positions. The derivation is straightforward, and again is left to the appendix.

### 3.2 Cumulative voting (CV)

With voting costs, the game cannot be reduced to the two party leaders. It now has three stages: a nomination stage, when the leaders choose the number of candidates; a turnout stage, when, after observing privately the realization of the voting cost, each voter decides whether or not to vote; and finally a voting stage, when voters at the polls choose how to cast their votes.

We focus on a semi-symmetric perfect Bayesian equilibrium such that within each party, all voters follow the same strategy. We denote by \( x_{-ip} \) the profile of votes cast by voters other than \( i \) who turned out and belong to \( p \). The equilibrium is a pair of party lists \( \{g, G\} \), a pair of cost thresholds \( \{c_M, c_m\} \), and a pair of voting profiles \( \{x_M, x_m\} \) such that: (i) at the voting stage, conditional on voting, voter \( i \) in party \( p \) sets \( x_{ip}(G, g, c_M, c_m, x_{-ip}, x_{p'}) \) so as to maximize the expected number of positions won by \( p \); (ii) at the turnout stage, all \( i \in p \) with \( c_i < c_p(G, g, c_{p'}, x_M, x_m) \) strictly prefer to vote, and all \( i \in p \) with \( c_i > c_p(G, g, c_{p'}, x_M, x_m) \) strictly prefer to abstain; and (iii) at the nomination stage, the two party leaders set \( g(G, x_M, x_m, c_M, c_m) \) and \( G(g, x_M, x_m, c_M, c_m) \) so as to maximize their party’s expected number of positions.\(^{15}\) For any positive turnout, if \( g < K \), party \( M \) is guaranteed \( \min[G, K - g] \) seats, and similarly, if \( G < K \), party \( m \) is guaranteed \( \min[g, K - G] \) seats. The positions contested are \( \max[0, g + G - K] \). We assume that non-contested positions are assigned to

\(^{15}\)Note that party leaders influence turnout indirectly through the number of candidates, recalling models of leaders’ enforced social norms in voting (Levine and Mattozzi, 2020).
candidates nominated by the parties even in the absence of voters’ turnout.

The voters’ turnout decision complicates the characterization of the equilibrium. The following lemma establishes some limited results that help to pin down the theoretical predictions for the parameterizations we use in the lab.

**Lemma 1.** (i) In all equilibria of the CV game with costly voting, \( g + G > K \).

(ii) If in equilibrium \( g + G = K + 1 \), then \( x_{i,m}^k = K/g \) and \( x_{i,M}^k = K/G \).

(iii) If in equilibrium \( g + G = K + 2 \) and \( g = G > 2 \), then \( x_{i,m}^k = K/g \) and \( x_{i,M}^k = K/G \).

The first two statements in the lemma are immediate. (i) If \( g + G \leq K \), there are no contested seats. With no contested seats, no voter with positive voting costs goes to the polls. Because non-contested positions are ensured, increasing the number of party candidates cannot cost any seat and may bring more party voters to the polls, and hence result in more victories. It is weakly dominant for either party to increase the number of candidates. (ii) If \( g + G = K + 1 \), there is a single contested seat. The competition between the two parties is over the weakest of their respective candidates. Strengthening the weakest candidate requires sharing votes equally, and each voter at the polls will vote accordingly. The proof of the third statement is less intuitive and we leave it to the appendix.

Distributing votes equally is an easy default for the voters, but we focus on equilibria with equal spreading of votes for two additional reasons as well. It seems plausible that such behavior would be the point of convergence of long experiences with CV when parties play fully their coordinating roles: for instance, Sawyer and MacRae (1962) and Goldburg (1994) document that equal spreading of votes over all party candidates was the norm in the Illinois State House, where, with more than a century of experience, behavior likely converged to incorporate CV’s lessons. Second, the explicit constraint that votes must be spread equally is part of a modified CV rule (“Equal and even CV”) applied in elections in Peoria, IL and at times proposed, because of its simplicity, as a possible model for wider adoption.\(^{16}\)

In what follows, we discuss the derivation of the equilibrium cost thresholds, and thus turnout, and then present the theory’s numerical predictions for a set of parameters that includes those we use in the experiment. In deriving the equilibrium cost thresholds, we conjecture an equilibrium with equal spreading of votes: \( x_{i,m}^k = K/g \), \( x_{i,M}^k = K/G \). All solutions we identify satisfy either

\(^{16}\)See for example the discussion by fairvote.org in http://archive.fairvote.org/factshts/comparis.htm
condition (ii) or condition (iii) in the lemma. Thus equal spreading of votes is an equilibrium strategy and the solutions are equilibria.

For given \( g \) and \( G \), equilibrium cost thresholds continue to trade off costs of voting and expected utility gains from influencing the election. As before, a voter may break an existing tie or cause a tie, but if the party’s candidates are fewer than the number of seats, by casting more than a single vote on each, the voter may also move the outcome from a loss to a win of all contested positions. Consider the problem for \( i \in M \). By voting, \( i \) breaks a tie if \( (K/G)S_{M-i} = (K/g)S_m \), or \( S_{M-i} = S_m(G/g) \); \( i \) causes a tie if \( (K/G)(S_{M-i} + 1) = (K/g)S_m \), or \( S_{M-i} = S_m(G/g) - 1 \). In addition, voter \( i \) can shift \( M \) from losing to winning all contested positions if both \( (K/G)S_{M-i} < (K/g)S_m \) and \( (K/G)(S_{M-i} + 1) > (K/g)S_m \), or \( S_{M-i} \in (S_m(G/g) - 1, S_m(G/g)) \). Denoting by \( \pi^T_p \) the probability that the votes of a member of party \( p \) move party \( p \) from losing to winning all contested positions, if \( c_M \in (0, 1) \) and \( G + g > K \), \( c_M \) must solve:

\[
c_M = [u(G) - Eu^{CV}_{T,M}(G, g)]\pi^T_M + [Eu^{CV}_{T,M}(G, g) - u(K - g)]\pi^{T-1}_M + [u(G) - u(K - g)]\pi^W_M
\]

where:

\[
Eu^{CV}_{T,M}(G, g) = \sum_{x=0}^{G} u(x) \left( \frac{G}{x} \right) \left( \frac{g}{K - x} \right) = \frac{G}{g + G}V.
\]

Or:

\[
c_M = \frac{V(g + G - K)}{K} \left[ \frac{G}{g + G}\pi^T_M + \frac{g}{g + G}\pi^{T-1}_M + \pi^W_M \right]
\]

The problem is analogous for minority voters. The equilibrium condition for an interior threshold \( c_m \) is:

\[
c_m = \frac{V(g + G - K)}{K} \left[ \frac{g}{g + G}\pi^T_m + \frac{G}{g + G}\pi^{T-1}_m + \pi^W_m \right]
\]

The pivot probabilities and the probabilities of winning different numbers of position in case of ties can be derived as under MP, taking into account that the number of candidates, in each party, may differ from the number of seats. We leave them to the appendix.

Given (5) and (6), and \( x^k_{im} = K/g \), \( x^k_{im} = K/G \), we can find party leaders’ optimal choice of \( G \) and \( g \). Given \( G \) and \( g \), we can verify that Lemma 1 applies, \( x^k_{im} = K/g \), \( x^k_{im} = K/G \) are indeed best responses, and the solution is an equilibrium.

13
Figure 1: Expected turnout rates and share of minority seats, MP and CV. The thick lines correspond to $c_M$, the thin lines to $c_m$; the bars correspond to the expected share of minority seats. $F$ is uniform over $[0, 1]$; $V = 4$.

3.3 Equilibria for the experimental parametrizations

Figure 1 shows the equilibrium turnout rates in the two parties, and the expected fraction of seats won by the minority under the two voting systems. The first column corresponds to MP, the second and third to CV (for $K = 2$ and $K = 4$, respectively. Recall that $K$ does not affect outcomes under MP). In each panel, the horizontal axis corresponds to different values of $M$, while upper and lower panels refer to different relative sizes of the two parties.\textsuperscript{17}

The figure highlights two main regularities. First, CV consistently increases the relative turnout of the minority: whether the relative size of the two parties is large or small, whether the number of open seats is just enough for CV to differentiate itself from MP ($K = 2$), or is higher ($K = 4$), the ratio $c_m/c_M$ is higher under CV than under MP. This remains true whether the electorate is small or large, unless the difference in size of the two parties becomes negligible (for $M = m + 1$ and large $M$), in which case turnout equalizes for the two

\textsuperscript{17}We found a unique equilibrium in all cases. Recall that with $F$ uniform, $c_p$ corresponds to the expected turnout rate in party $p$. We discuss in the appendix the surprising lack of a consistent underdog effect ($c_m > c_M$) in the MP model. For both MP and CV, raising $K$ to 6 does not change the qualitative results.
parties under both MP and CV. Second, the expected fraction of seats won by the minority is consistently higher under CV. The effect is most striking when the minority is relatively small \((M = 2m)\), and its expected share of seats never rises above 14% under MP, less than half its share of the electorate, as opposed to being consistently close to 40% under CV.

In all cases, the minority party sets \(g < K\) under CV, and thus exploits the possibility to cumulate votes. In the figure, when \(K = 2\), \(g = 1\) for all \(M\) and \(m\) (while \(G = K = 2\)); when \(K = 4\), \(g = 2\) if \(m = M/2\) and \(g = 3\) otherwise (while \(G = 3\) always). When \(G = K\), the minority’s cumulation of votes results in a higher probability of affecting the outcome, incentivizing turnout; when \(G < K\), the difference in turnout probabilities is reduced, but the share of minority victories is boosted by the seats left uncontested by the majority. Thus in the lower right panel, \(g = G = 3\), and the two parties’ turnout probabilities converge, but the uncontested seat keeps the share of expected minority victories higher than under MP.

The simulations generate the hypotheses we test in the experiment, and thus we focus on small size electorates. But how would the voting rules compare when the electorates are large? For MP, given its correspondence to single winner plurality systems, the theoretical predictions are known: turnout rates fall with the increase in population, but less so for the minority, whose probability of success increases with population size (for given population share) while remaining below 50 percent (Levine and Palfrey, 2007; Herrera et al., 2014). There is no corresponding theoretical analysis of turnout in large populations under CV. However, the regularities we see in our simulations match the theoretical results found by Herrera et al. for proportional representation: when the difference in size between the two groups persists in large electorates, the ratio of minority/majority turnout rates is consistently higher under proportional representation than under plurality. In our simulations, we observe the same results under CV when \(M = 2m\), for larger values of \(M\). As mentioned earlier, the logic behind the turnout decision is different under proportional representation and CV, but in both cases the comparison to MP reflects the smaller impact of a large electorate on the minority decision, because the marginal impact of an additional vote is larger in a smaller group in the case of proportional representation, or because of the positive impact of cumulated votes on pivotality in the

\[^{18}\text{We have also run a few additional simulations with } M = 30 \text{ and } M = 40, \text{ confirming the qualitative results.}\]
Finally, we can compare the results to minority victories in the absence of voting costs, and thus of turnout effects. Under MP, as we know, the minority never wins any seat, as opposed to the small but positive share predicted with voting costs. Under CV and costless voting, the expected share of minority victories is $1/2$ if $m = M - 1 > K/2$, and either $1/3$ (if $K = 2$) or $1/4$ (if $K = 4$) if $m = M/2$. Accounting for turnout thus softens the impact of relative party size, reducing expected minority victories for $m = M - 1$, but increasing them for $m = M/2$. Under CV, the minority achieves substantive representation but is always expected to maintain its minority status in the allocation of seats.

4 The Experiment

The experiment reproduces exactly the theoretical model. Our main focus is the comparison of turnout rates, for both parties, and the fraction of minority victories under the two voting rules, MP and CV. To evaluate the robustness of the results and to test the power of the theoretical framework we implemented four different parametrizations: while we kept $M = 4$ throughout the experiment, we varied $m$ between 2 and 3; for each $m$, we set $K = 2$ and $K = 4$. In all treatments, voting costs were drawn independently across participants from a uniform distribution with support $[0, 100]$, and $V$, the value of controlling all positions, was set at 400.

The number of candidates fielded by each party under CV was set at the theoretically optimal value for each parameterization and we constrained voters who turned out to spread their votes equally over their party candidates. Not only is equal spreading of votes part of the equilibrium, but in the absence of distinguishing features among candidates or seats and of any communication among voters, variations in the distribution of votes would simply reflect noise. Participants acted as eligible voters: at each round, each drew an independent voting cost and decided whether or not to vote. The design thus mimics the numerical simulations, with $M = 4$. We reproduce it in Table 1, together with the theoretical predictions.

The table dictates the hypotheses to test. The first set concerns turnout

---

19Casella and Gelman (2008) study pivot probabilities in large electorates when voters can cast more than a single vote. The problem analyzed—simultaneous referenda over multiple binary decisions—is different, but the effect of cumulation on pivot probabilities seems likely to generalize.

16
Table 1: Experimental Design and Predictions. F uniform over [0,100]; V = 400.

<table>
<thead>
<tr>
<th>M, m</th>
<th>K</th>
<th>Rule</th>
<th>G, g</th>
<th>$c_m$</th>
<th>$c_M$</th>
<th>($c_m / c_M$)</th>
<th>Exp. Min. Seats</th>
<th>Exp. Min. Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>4, 2</td>
<td>2</td>
<td>MP</td>
<td>2, 2</td>
<td>0.54</td>
<td>0.66</td>
<td>0.81</td>
<td>0.22</td>
<td>0.11</td>
</tr>
<tr>
<td>4, 2</td>
<td>2</td>
<td>CV</td>
<td>2, 1</td>
<td>0.67</td>
<td>0.30</td>
<td>2.23</td>
<td>0.84</td>
<td>0.42</td>
</tr>
<tr>
<td>4, 2</td>
<td>4</td>
<td>MP</td>
<td>4, 4</td>
<td>0.54</td>
<td>0.66</td>
<td>0.81</td>
<td>0.44</td>
<td>0.11</td>
</tr>
<tr>
<td>4, 2</td>
<td>4</td>
<td>CV</td>
<td>3, 2</td>
<td>0.42</td>
<td>0.30</td>
<td>1.41</td>
<td>1.52</td>
<td>0.38</td>
</tr>
<tr>
<td>4, 3</td>
<td>2</td>
<td>MP</td>
<td>2, 2</td>
<td>0.56</td>
<td>0.79</td>
<td>0.71</td>
<td>0.24</td>
<td>0.12</td>
</tr>
<tr>
<td>4, 3</td>
<td>2</td>
<td>CV</td>
<td>2, 1</td>
<td>0.49</td>
<td>0.27</td>
<td>1.84</td>
<td>0.84</td>
<td>0.42</td>
</tr>
<tr>
<td>4, 3</td>
<td>4</td>
<td>MP</td>
<td>4, 4</td>
<td>0.56</td>
<td>0.79</td>
<td>0.71</td>
<td>0.48</td>
<td>0.12</td>
</tr>
<tr>
<td>4, 3</td>
<td>4</td>
<td>CV</td>
<td>3, 3</td>
<td>0.53</td>
<td>0.54</td>
<td>0.98</td>
<td>1.68</td>
<td>0.42</td>
</tr>
</tbody>
</table>

rates, \( \{c_m, c_M\} \). First, in all parametrizations, the relative minority/majority turnout \( (c_m / c_M) \) is strictly higher under CV than under MP (H1). Second, under MP the turnout rate is higher for the majority; under CV it is higher for the minority in three of the four parametrizations, and barely lower in the fourth (H2). The second set of hypotheses concerns expected minority victories. First, in all parametrizations the expected share of seats won by the minority is higher under CV than under MP (H3). Second, as a check of the theoretical model, the share varies with the voting rule but for each rule is very close to constant across parametrizations (H4).

We conducted the experiment between August and October 2020, with participants recruited using the Columbia Experimental Laboratory for the Social Sciences (CELSS)’ ORSEE website\(^{20}\). Most subjects were undergraduate students at Columbia University or Barnard College. All sessions were online due to the COVID-19 pandemic: participants received instructions and communicated with experimenters using the Zoom videoconferencing software, and accessed the experiment interface on their personal computer’s web browser. The experiment was programmed in z-Tree (Fischbacher, 2007) and run virtually using z-Tree unleashed (Duch et al., 2020). Each experimental session lasted about 90 minutes with average earnings of $23. With the exception of a more visual style for the instructions, the experiment developed very similarly to in-person experiments in the lab.\(^{21}\)

\(^{20}\)Greiner (2015).
\(^{21}\)Online appendix 2 contains a reproduction of the instructions.
During each session, party sizes were kept fixed, and participants played 15 consecutive rounds each of four treatments, CV and MP for each of $K = 2$ and $K = 4$. Party affiliations were kept constant within each treatment to facilitate learning but were assigned randomly across treatments. In each round, two groups were formed randomly, each composed of $m$ minority and $M$ majority members. At the end of the round, an outcome screen reported the party affiliations of the $K$ winning candidates and the number of members of each party who had voted. Each participant’s final earnings corresponded to the sum of their earnings from one randomly drawn round from each treatment (in addition to the $5 show-up fee). For given $m$, either 2 or 3, we ran two experimental sessions for each of four orders of treatments. Thus eight sessions were conducted with $m = 2$ (12 subjects per session), and eight with $m = 3$ (14 subjects per session), for a total of 208 experimental subjects.

5 Experimental Results

Figure 2 reports the turnout rates for minority and majority voters in the different treatments, in the upper panels, and the two parties’ relative rate in the lower panel. For all experimental values of $K$ and $m$, minority turnout is higher under CV than under MP. The effect is particularly strong for $m = 2$, but remains consistently positive, if more muted, with $m = 3$ as well. The difference remains positive even when theory predicts instead a decline in turnout. Interestingly, with one exception, we do not observe the predicted decline in turnout for the majority under CV. The exception is $m = 3$ and $K = 2$, the parametrization for which the theory predicts the sharpest fall in majority turnout. With these parameters, the majority is certain of one victory but, with a relatively large minority concentrating all votes on a single candidate, the chances of a second majority victory are low.

The lower panel shows the final effect on relative turnout frequencies: in all cases, the minority’s relative turnout rate increases under CV. The theory also predicts that the minority’s turnout rate should be lower than the majority’s in all MP treatments, higher in three of the four CV treatments, and barely lower

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22The orders were: 1: \{CVK2, MPK2, MPK4, CVK4\}, 2: \{MPK2, CVK2, CVK4, MPK4\}, 3: \{MPK4, CVK4, CVK2, MPK2\}, 4: \{CVK4, MPK4, MPK2, CVK2\}.  
23In all figures below, reported theoretical predictions are calculated on the basis of the realized experimental cost draws.
in the fourth. The comparative prediction is again satisfied in all cases.

The evidence from the figure is confirmed by the statistical analysis. In the first two columns of Table 2, we regress subjects’ turnout decisions on each subject’s voting cost realization and on a series of dummy variables, reflecting $K$, $m$, the voting system, and all cross effects among the three variables, controlling for the round number, and the order of treatments in the session. The Table reports the estimation of a linear probability model; results from a probit model are fully consistent and are reported in the online appendix.

The first column refers to the minority. Under MP, the increase in the minority’s relative size from $m = 2$ to $m = 3$ increases turnout, and so does the increase in the number of seats from $K = 2$ to $K = 4$, although the two effects are muted when they occur together. The most interesting result is the strongly positive effect of CV, especially but not only when $m$ is small and $K$ is small. The order of treatments has no detectable effect, while experience (or fatigue) at
Table 2: Turnout and Minority Victories.

<table>
<thead>
<tr>
<th></th>
<th>Turnout: Minority</th>
<th>Turnout: Majority</th>
<th>% Seats: Minority</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>K = 4</strong></td>
<td>0.184</td>
<td>0.002</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>(0.052)</td>
<td>(0.033)</td>
<td>(0.026)</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.049]</td>
<td>[0.065]</td>
</tr>
<tr>
<td><strong>m = 3</strong></td>
<td>0.171</td>
<td>0.083</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>(0.052)</td>
<td>(0.044)</td>
<td>(0.028)</td>
</tr>
<tr>
<td></td>
<td>[0.001]</td>
<td>[0.064]</td>
<td>[0.001]</td>
</tr>
<tr>
<td><strong>CV</strong></td>
<td>0.236</td>
<td>0.008</td>
<td>0.158</td>
</tr>
<tr>
<td></td>
<td>(0.061)</td>
<td>(0.035)</td>
<td>(0.025)</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.810]</td>
<td>[0.000]</td>
</tr>
<tr>
<td><strong>(K = 4) × (m = 3)</strong></td>
<td>-0.199</td>
<td>0.048</td>
<td>-0.122</td>
</tr>
<tr>
<td></td>
<td>(0.069)</td>
<td>(0.048)</td>
<td>(0.040)</td>
</tr>
<tr>
<td></td>
<td>[0.004]</td>
<td>[0.318]</td>
<td>[0.002]</td>
</tr>
<tr>
<td><strong>(K = 4) × CV</strong></td>
<td>-0.087</td>
<td>0.020</td>
<td>0.041</td>
</tr>
<tr>
<td></td>
<td>(0.082)</td>
<td>(0.045)</td>
<td>(0.033)</td>
</tr>
<tr>
<td></td>
<td>[0.291]</td>
<td>[0.657]</td>
<td>[0.215]</td>
</tr>
<tr>
<td><strong>(m = 3) × CV</strong></td>
<td>-0.149</td>
<td>-0.137</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>(0.078)</td>
<td>(0.052)</td>
<td>(0.037)</td>
</tr>
<tr>
<td></td>
<td>[0.058]</td>
<td>[0.009]</td>
<td>[0.308]</td>
</tr>
<tr>
<td><strong>(K = 4) × (m = 3) × CV</strong></td>
<td>0.065</td>
<td>0.082</td>
<td>-0.019</td>
</tr>
<tr>
<td></td>
<td>(0.104)</td>
<td>(0.066)</td>
<td>(0.048)</td>
</tr>
<tr>
<td></td>
<td>[0.537]</td>
<td>[0.218]</td>
<td>[0.695]</td>
</tr>
<tr>
<td><strong>Treatment order=2</strong></td>
<td>-0.024</td>
<td>0.033</td>
<td>-0.012</td>
</tr>
<tr>
<td></td>
<td>(0.044)</td>
<td>(0.045)</td>
<td>(0.016)</td>
</tr>
<tr>
<td></td>
<td>[0.579]</td>
<td>[0.456]</td>
<td>[0.441]</td>
</tr>
<tr>
<td><strong>Treatment order=3</strong></td>
<td>0.044</td>
<td>0.025</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>(0.043)</td>
<td>(0.045)</td>
<td>(0.016)</td>
</tr>
<tr>
<td></td>
<td>[0.304]</td>
<td>[0.581]</td>
<td>[0.505]</td>
</tr>
<tr>
<td><strong>Treatment order=4</strong></td>
<td>0.026</td>
<td>0.011</td>
<td>0.021</td>
</tr>
<tr>
<td></td>
<td>(0.046)</td>
<td>(0.049)</td>
<td>(0.017)</td>
</tr>
<tr>
<td></td>
<td>[0.574]</td>
<td>[0.824]</td>
<td>[0.221]</td>
</tr>
<tr>
<td><strong>Round</strong></td>
<td>-0.007</td>
<td>-0.005</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
<td>[0.392]</td>
</tr>
<tr>
<td><strong>Voting Cost</strong></td>
<td>-0.008</td>
<td>-0.008</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
<td>[0.000]</td>
</tr>
<tr>
<td><strong>Constant</strong></td>
<td>0.862</td>
<td>1.039</td>
<td>0.092</td>
</tr>
<tr>
<td></td>
<td>(0.049)</td>
<td>(0.045)</td>
<td>(0.023)</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
<td>[0.000]</td>
</tr>
<tr>
<td><strong>Observations</strong></td>
<td>4800</td>
<td>7680</td>
<td>1920</td>
</tr>
<tr>
<td><strong>R²</strong></td>
<td>0.276</td>
<td>0.246</td>
<td>0.148</td>
</tr>
</tbody>
</table>

Regressions follow the specification:

\[ Y_{i,r} = \beta_0 + \beta_1 T_r + \beta_2 X_{i,r} + \epsilon_{i,r} \]

where \( T \) is a vector with dummies for the three treatment effects (\( m = 3, CV, K = 4 \)) and all their interactions, by round, and \( X \) is the vector of controls reported in the bottom half of the table. \( Y_{i,r} \) is a binary \( \{0, 1\} \) variable. In columns 1 and 2, it captures voting in round \( r \) for individual \( i \), in column 3, the event of a minority victory in round \( r \) in group \( i \). Standard errors for columns 1 and 2 are clustered at the individual level. Standard errors for column 3 are clustered at the \( (M+m) \) group level. The excluded case is \( m = 2, K = 2, MP \), with treatment order \( \{CVK2, MPK2, MPK4, CVK4\} \).

Standard errors are reported in parentheses, above p-values, reported in brackets.
Figure 3: CDF’s of Cost Cutpoints. The dotted lines correspond to MP; the solid lines to CV.

later experimental rounds causes a significant but quantitatively small decline in turnout.

The second column refers to the majority. Under MP, the regression finds a small and only marginally significant increase in turnout with \( m = 3 \). Under CV, there is a decline in turnout in the \( m = 3, K = 2 \) case, but majority turnout remains otherwise relatively constant, regardless of the voting system or the values of \( K \) and \( m \). For the majority too we find no effect of treatment order and a small significant decline in turnout at later rounds.

Summarizing then, CV increases the minority’s turnout, both in absolute terms and relative to the majority, supporting both hypotheses H1 and H2. The most noticeable deviation from the theory is the majority’s persistently high turnout under CV.\(^\text{24}\)

Figure 2 invites a natural question: CV affects positively the minority’s propensity to vote, but does the aggregate effect mirror a widespread change in behavior, or the changed attitude of a few outliers? Figure 3 addresses the question by reporting the CDF’s of the individual cutpoints estimated from our data.\(^\text{25}\)

\(^{24}\) As a result, total turnout, predicted to fall under CV, remains in fact constant in the lab. We discuss the data on total turnout in the online appendix.

\(^{25}\) See the online appendix for details on the estimation, as well as additional information on individual behavior.
The minority’s higher propensity to vote under CV operates throughout the distribution of individual cutpoints. Especially with \( m = 2 \), when the minority is half the majority, the move to CV causes an unambiguous shift rightward of the whole distribution: the minority cutpoints distribution under CV FOSD’s the distribution under MP. The effect is particularly strong when \( K = 2 \), and under CV the minority cumulates all votes on a single candidate (\( g = 1 \)): CV makes the prospect of winning one seat realistic and encourages participation. When \( m = 3 \) and the difference in size between the two parties is smaller, the minority still shifts homogeneously towards higher turnout but the move is less pronounced.

As for the majority, with \( m = 2 \), majority members barely modify their propensity to vote. With \( m = 3 \), the conclusion is similar when \( K = 4 \); when \( K = 2 \), however, the majority’s decline in turnout noted earlier comes from a consistent decline in voting throughout the majority’s cutpoint distribution: the distribution under MP FOSD’s the distribution under CV. As remarked earlier, it is the only case in which the predicted decline in majority’s turnout is seen in the data.

Figure 3 also shows the heterogeneity in behavior we see in the lab. Under both voting rules and for all parametrizations, the theoretical CDF’s have a single step at the equilibrium cutpoint; in line with previous results from similar experiments\(^26\), this is not what we see in the data.

Did CV help the minority secure more seats? Figure 4 shows that the answer is positive.

For every parametrization, CV increases the fraction of minority victories, and does so very significantly, a result confirmed by the third column of Table 2.\(^27\) The data strongly support hypothesis H3. Note that, as predicted, the share of minority victories remains under 50% in all cases.

As for hypothesis H4, supported too is the prediction that the differences in minority victories should be larger across voting rules, for given parametrization, than across parametrizations, for given voting rule. The point prediction of constant share of minority victories across parametrizations, for given voting rule, fares slightly less well, with the largest exception the fewer than expected successes of the minority under CV when \( K = 2 \) and \( m = 2 \), reflecting the

\(^{26}\) For example, Levine and Palfrey (2007).

\(^{27}\) As expected, minority victories also increase with the minority size (\( m = 3 \)). In these regressions again there is no effect of treatment order, and neither is there a change from earlier to later rounds, reflecting the previous finding of a parallel and equally small decline in turnout for both parties as the sessions proceeded.
Figure 4: *Share of seats won by the minority.* Blue columns correspond to the data, grey to the theory. The 95% CI’s are calculated from standard errors clustered at the level of the voting group.

unexpected high turnout of the majority in this treatment. Even in this case, however, the share of minority seats is more than double what it is under MP.

6 Conclusions

As we write, debates over voting rights rage in Congress, in state legislatures, in the courts, in the media. Initiatives aimed at limiting access to the polls and the prospect of partisan redistricting following the 2020 census increase fears of disenfranchisement. Encouraging minority turnout and making the political process more robust to gerrymandering are high priorities. Voting rules like CV have the potential to help both. Although much remains to be studied, in the lab, such potential is fulfilled.

7 Appendix

7.1 CV when voting is costless

**Proposition 1.** In the absence of voting costs, in all party-optimal equilibria of the CV voting game: (i) for all $m < M/K$, the minority never wins any seat;
(ii) for all \( m \geq M/K, M > m, \) and \( K \):

\[
 z \in \left( \frac{Km - M}{M + m}, \frac{Km + m}{M + m} \right) \quad \text{if} \quad \frac{Km - M}{M + m} \notin \mathbb{Z}
\]

\[
 z = \begin{cases} 
 \frac{Km - M}{M + m} & \text{with prob } m/(m + M) \\
 \frac{Km + m}{M + m} & \text{with prob } M/(m + M)
\end{cases} \quad \text{if} \quad \frac{Km - M}{M + m} \in \mathbb{Z}
\]

Because \( (Km + m)/(M + m) - (Km - M)/(M + m) = 1 \), if \( \frac{Km - M}{M + m} \) is not an integer, there exists a unique integer value of \( z \) in the relevant interval; if \( \frac{Km - M}{M + m} \) is an integer, then \( \frac{Km + m}{M + m} \) is an integer as well.

**Proof.** We establish the proposition by proceeding in three steps.

1. First, we note that the identity of purpose between party leaders and voters implies that in all party-optimal equilibria we can think of the party leaders as controlling not only the number of party candidates but also the distribution of votes cast by party voters.

2. Second, we show that the proposition identifies the number of seats won by the minority when both parties follow maximin strategies.

   (i) Suppose first that \( m < M/K \). Then the \( M \) party can guarantee itself all \( K \) seats by dividing its votes equally over \( K \) candidates, and the \( m \) party cannot win any seat.

   (ii) Suppose then \( m > M/K \). For any \( x_m \), party \( m \) maximizes the probability of winning \( z \) seats by dividing its votes equally over \( z \) candidates, and guarantees itself \( z \) seats if \( mK/z > MK/(K - z + 1) \), or \( z < (Km + m)/(M + m) \). At the same time, party \( M \) maximizes the probability of winning \( (K - z) \) seats by dividing its votes equally over \( K - z \) candidates, and guarantees itself \( K - z \) seats if \( MK/(K - z) > mK/(z + 1) \), or \( z > (Km - M)/(M + m) \). We require \( z \) to be an integer.

   (ii.a) Suppose first that \( (Km - M)/(M + m) \) is not an integer. Since \( (Km + m)/(M + m) - (Km - M)/(M + m) = 1 \), if \( (Km - M)/(M + m) \) is not an integer, then there exists a unique integer value of \( z \in \left( \frac{Km - M}{M + m}, \frac{Km + m}{M + m} \right) \).

   (ii.b and iii) Finally, suppose that either \( m > M/K \) and \( (Km - M)/(M + m) \) is an integer, or \( m = M/K \) (and thus \( (Km - M)/(M + m) = 0 \)). Then the \( m \) party can guarantee itself \( (Km - M)/(M + m) \equiv z \) seats, but can do better by spreading votes equally over \( (Km - M)/(M + m) + 1 = (Km + m)/(M + m) \equiv z \) candidates. Similarly, the \( M \) party can guarantee itself \( K - z \) seats, but can do better by spreading votes equally over \( K - z + 1 = K - z \) candidates.

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In equilibrium then, party $m$ ($M$) spreads its votes equally over $\pi$ ($K - \pi$) candidates; a total of $K + 1$ candidates receive votes, and all are tied with $\lfloor K/(K+1) \rfloor (M + m)$ votes each. The tie-break rule selects $K$ winners randomly from the $K + 1$ candidates. It then follows that:

$$\text{prob}(z = \pi) = \frac{K - \pi}{K + 1} = 1 - \frac{\pi}{K + 1} = \frac{M}{m + M}$$

$$\text{prob}(z = \pi) = 1 - \text{prob}(z = \pi) = \frac{m}{m + M}$$

3. Because $u(k)$ is linear in $k$, the game is constant-sum. It follows that party-optimal equilibria are equilibria of a constant sum, two-player game. Hence, if equilibria exist, they must all yield maximin payoffs. It is not difficult to verify that the strategies described above are equilibria: neither part has a profitable deviation. Thus equilibria exist and all yield $z$ minority victories. □

### 7.2 Costly voting

#### 7.2.1 Multi-winner plurality (MP). Pivot probabilities and probabilities of winning seats

We report here the binomial formulas for the pivot probabilities. Under MP, such formulas are well-known (see for example Levine and Palfrey, 2007).

$$\pi_m^{T-1} = \sum_{x=0}^{m-1} \binom{m - 1}{x} \binom{M}{x + 1} F(c_m)^x [1 - F(c_m)]^{m-1-x} F(c_M)^{x+1} [1 - F(c_M)]^{M-x+1}$$

$$\pi_m^{T} = \sum_{x=0}^{m-1} \binom{m - 1}{x} \binom{M}{x} F(c_m)^x [1 - F(c_m)]^{m-1-x} F(c_M)^{x} [1 - F(c_M)]^{M-x}$$

and:

$$\pi_M^{T-1} = \sum_{x=1}^{m} \binom{m}{x} \binom{M - 1}{x - 1} F(c_m)^x [1 - F(c_m)]^{m-x} F(c_M)^{x-1} [1 - F(c_M)]^{M-1-(x-1)}$$

$$\pi_M^{T} = \sum_{x=0}^{m} \binom{m}{x} \binom{M - 1}{x} F(c_m)^x [1 - F(c_m)]^{m-x} F(c_M)^{x} [1 - F(c_M)]^{M-1-x}$$

The frequency of minority victories is sensitive to the relative turnout rates of the two parties, captured by the ratio of the two thresholds, $c_m/c_M$. Al-
though the study of costly voting models has identified an “underdog effect”—
the tendency for the minority’s turnout rate to be higher than the majority’s,
or $c_m > c_M$—the existence of such an effect is sensitive to the exact specification
of the model. It has been proven in a number of scenarios: when the voting cost
is fixed and equal for all (Taylor and Yildirim, 2010a); when voters’ direction
of preferences is randomly drawn (Ledyard, 1984; Taylor and Yildirim, 2010b);
when the size of the electorate is uncertain (Herrera et al., 2014; Krishna and
Morgan, 2015). The specification used here differs from these models, and relative
turnout under MP depends on $V$, the value of winning all seats. Because
the model is widely used but this observation is missing from the literature, we
make it explicit in the following remark.

**Remark.** For any finite $M$, $m$, and $F$ there exists a finite $\hat{V}(M,m)$ such
that if $V = \hat{V}$, then $c_m = c_M$.

**Proof.** Call $\hat{c}$ the median of $F(c)$. Straightforward manipulations of the
pivot probabilities show that if $c_m = c_M = \hat{c}$, and thus $F(c_m) = 1 - F(c_M) = 1/2$,
then $(\pi^T_m + \pi^T_M - 1) = (\pi^T_M + \pi^T_M - 1) = (1/2)^{M+m-1} \binom{M + m}{m}$. Hence for any $M$
and $m$, $c_m = c_M = \hat{c}$ is an equilibrium as long as $\hat{c} = (V/2)(1/2)^{M+m-1} \binom{M + m}{m}$,
or $V = \hat{c} \left( \frac{2^{(M+m)}}{\binom{M + m}{m}} \right) = \hat{V}$. □

Note that our focus is not on the two parties’ relative turnout per se, but
on the impact on such relative turnout of the voting rule.

The derivation of the probabilities of winning different numbers of seats is
straightforward. Consider the problem from the perspective of a minority voter.
Begin with the probability of losing all positions, $\Pr(W_m = 0)$. Such probability
equals the probability that either all minority candidates receive strictly fewer
votes than the majority candidates, or that all candidates are tied but minority
candidates lose all tie-breaks. Or, $\Pr(W_m = 0) = \Pr(S_m < S_M) + \Pr((S_m =
\(S_m \cap (m \text{ loses all tie-breaks})\]. That is:

\[
\Pr(W_m = 0) = \\
= \sum_{S_m=0}^{M} \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \sum_{S_m=S_m+1}^{m} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} + \\
+ \sum_{S_m=0}^{M} \binom{M}{S_m} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \times \\
\times F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} \left(1/ \left(\binom{2K}{K}\right)\right)
\]

Similarly, the probability that \(m\) wins all positions, \(\Pr(W_m = K)\) equals the probability that either all minority candidates receive strictly more votes than the majority candidates, or that all candidates are tied but minority candidates win all tie-breaks. Or, \(\Pr(W_m = K) = \Pr(S_m > S_M) + \Pr[(S_m = S_M) \cap (m\text{ wins all tie-breaks})]\). That is:

\[
\Pr(W_m = K) = \\
= \sum_{S_m=0}^{m-1} \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \sum_{S_m=S_m+1}^{m} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} + \\
+ \sum_{S_m=0}^{M} \binom{M}{S_m} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \times \\
\times F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} \left(1/ \left(\binom{2K}{K}\right)\right)
\]

The probabilities of other numbers of minority victories can be derived in the same fashion. The probability of electing \(w\) minority candidates, with \(w \in (0, K)\) equals the probability that all candidates are tied and \(m\) wins \(w\) tie-breaks. Thus:

\[
\Pr(W_m = w) = \\
= \sum_{S_m=0}^{M} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} \times \\
\times \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \left(\binom{K}{w}\left(\frac{K}{K-w}\right) / \left(\frac{2K}{K}\right)\right)
\]

For given \(M, m, K, F(c)\), and \(\{u(k)\}\), the equilibrium yields expected
turnout rates for voters of the two parties, the probabilities of winning 0, 1, ..., \( K \) positions for each party, and ex ante expected utility for an \( M \) and an \( m \) voter.\(^{28}\)

### 7.2.2 Cumulative Voting (CV)

**Lemma 1.** (i) In all equilibria of the CV game with costly voting, \( g + G > K \).

(ii) If in equilibrium \( g + G = K + 1 \), then \( x^{k}_{i,m} = K/g \) and \( x^{k}_{i,M} = K/G \). (iii) If in equilibrium \( g + G = K + 2 \) and \( g = G > 2 \), then \( x^{k}_{i,m} = K/g \) and \( x^{k}_{i,M} = K/G \).

**Proof of claim (iii).** Suppose \( g + G = K + 2 \). We want to establish that spreading votes equally over all party candidates is a best response for a voter at the polls, facing a scenario where all other voters do so. Consider voter \( i \in m \).

Note that, ignoring \( i \)'s votes, all candidates of a given party are expected to have the same number of votes, for any turnout. By dividing votes unequally, voter \( i \) increases the chances of victory for the subset of party candidates receiving more than \( K/g \) of his votes, at the expense of the others. But \( g + G = K + 2 \):

only two positions are contested and it cannot be advantageous to withdraw votes from two or more party candidates. If \( i \) withdraws votes, it is from one candidate. Call such a candidate \( \tilde{k} \).

Suppose first \( x^{k}_{i,m} = 0 \): \( i \) casts 0 votes on \( \tilde{k} \). Either \( S_{m-1} K/g \) votes are enough for \( \tilde{k} \) to win a seat, or they are not. If \( \tilde{k} \) wins a seat, whether \( i \) distributes all her votes equally or unequally on the remaining \((g - 1)\) candidates is irrelevant: all \( m \) candidates have at least \( S_{m-1} K/g \) votes, and if \( S_{m-1} K/g \) votes are enough to win, \( m \) wins both contested seats. If instead \( S_{m-1} K/g \) votes are not enough for \( \tilde{k} \) to win a seat, then, with one fewer competitive candidate, only one position is contested: \( i \)'s most profitable deviation must be to strengthen \( m \)'s weakest candidate among the \((g - 1)\) remaining. Thus, whether \( \tilde{k} \) remains competitive or not, \( i \)'s best deviation is to spread votes equally over the remaining \((g - 1)\) candidates.

Can this be advantageous? It will be so if the shift in votes moves the candidates receiving more votes from a loss to either a tie or a victory. That is, if:

\[
\frac{S_{m-1} K}{g} + \frac{K}{g} \leq \frac{S_{M} K}{G} \quad \text{but} \quad \frac{S_{m-1} K}{g} + \frac{K}{g-1} \geq \frac{S_{M} K}{G}
\]

\(^{28}\)Note in particular that if \( u(K) - Eu_{M}^{MP} = Eu_{M}^{MP} - u(0) \), or \( Eu_{M}^{MP} = [u(K) - u(0)]/2 \), the equilibrium thresholds \( \{c_{m}, c_{M}\} \) are identical to the thresholds that solve the corresponding costly voting problem with a single winner.
Or, multiplying by \( g/K \):

\[
S_{m-i} + 1 < S_M(g/G) \quad \text{but} \quad S_{m-i} + g/(g-1) \geq S_M(g/G)
\]

Since \( g = G \), and \( g/(g-1) = 1 + 1/(g-1) \), the condition becomes:

\[
S_{m-i} + 1 < S_M \quad \text{but} \quad S_{m-i} + 1 + 1/(g-1) \geq S_M
\]

Both \( S_{m-i} \) and \( S_M \) are integers. If \( g > 2 \), \( 1/(g-1) < 1 \); hence the condition cannot be satisfied.

Note that we have not ruled out moving the targeted candidates from a tie to a victory. The reason is that such a shift, if it occurs, cannot increase utility. Suppose that if \( i \) spreads votes equally over all \( g \) candidates, all are tied for most votes with the \( G \) candidates. Each party is nominating \( K/2 + 1 \) candidates for \( K \) positions and \( Eu_{i,m} = V/2 \). Alternatively, if \( i \) shifts votes, \( K/2 \) candidates from party \( m \) receive the most votes and are sure to be elected, or, indicating by \( d \) the deviation \( Eu_{i,m}^d = V/2 = Eu_{i,m} \).

The argument shows that \( i \) cannot gain from withdrawing all her votes from candidate \( \tilde{k} \). It continues to hold, however, if \( i \) withdraws her votes from \( \tilde{k} \) only partially: for the reasons argued above, equal spreading of votes on all remaining \( (g-1) \) candidates must remain \( i \)'s optimal deviation, and if shifting \( K/g \) votes from \( \tilde{k} \) to the remaining \( g-1 \) candidates cannot improve their chances of winning, neither can shifting fewer than \( K/g \) votes.

Finally, note that all arguments apply identically to \( i \in M \), with obvious changes of notation. \( \square \)

Could we have ruled out the deviation a priori? If, at the polls, voter \( i \) found it profitable to spread votes equally over fewer than \( g \) candidates, wouldn’t \( m \)'s party leaders, who share voter \( i \)'s objective of maximizing party victories, nominate fewer candidates at the nomination stage? This line of reasoning neglects that party leaders’ nominations affect turnout through the cost thresholds, a consideration that is not shared by voters at the polls, who take turnout as given. In equilibrium, leaders’ nominations and voters’ targeting of candidates at the polls must coincide, but such coincidence cannot be assumed a priori.
7.2.3 Cumulative Voting (CV). Pivot probabilities and probabilities of winning seats

Consider first the perspective of a majority voter. The pivot probabilities correspond to the probabilities of the three events described in the text—breaking a tie (if \((K/G)S_{M-i} = (K/g)S_m\)), making a tie (if \((K/G)(S_{M-i} + 1) = (K/g)S_m\)), or moving the outcome from a loss to a win on all contested positions (if \(S_{M-i} \in (S_m(G/g) - 1, S_m(G/g))\)). Note that since \(S_{M-i}\) and \(S_m\) are non-negative integers, the first event is only possible if either \(G/g\) is an integer, or \(S_{M-i} = S_m = 0\); the second event is only possible if \(G/g\) is an integer, and the third event is only possible if \(G/g\) is not an integer.

The equations corresponding to the pivot probabilities are logically straightforward:

\[
\tilde{\pi}_M^T = I_Q[(G/g)S_m] \sum_{S_m=0}^{m} \left\{ \binom{m}{S_m} F(c_m)^{S_m}[1 - F(c_m)]^{m-S_m} \left( \frac{M - 1}{(G/g)S_m} \right)^{(G/g)S_m}[1 - F(c_M)]^{M - 1 - [(G/g)S_m - 1]} \right\}
\]

\[
\tilde{\pi}_M^{T-1} = I_Q[(G/g)S_m] \sum_{S_m=1}^{m} \left\{ \binom{m}{S_m} F(c_m)^{S_m}[1 - F(c_m)]^{m-S_m} \left( \frac{M - 1}{(G/g)S_m - 1} \right)^{(G/g)S_m-1}[1 - F(c_M)]^{M - 1 - [(G/g)S_m - 1]} \right\}
\]

and

\[
\tilde{\pi}_M^W = (1 - I_Q[(G/g)S_m]) \sum_{S_m=0}^{m} \left\{ \binom{m}{S_m} F(c_m)^{S_m}[1 - F(c_m)]^{m-S_m} \left( \frac{M - 1}{[(G/g)S_m]} \right)^{(G/g)S_m}[1 - F(c_M)]^{M - 1 - [(G/g)S_m]} \right\}
\]

where \(I_Q[(G/g)S_m] = 1\) if \((G/g)S_m\) is an integer, and 0 otherwise, and \(\lfloor x \rfloor\) is the floor function, denoting the greatest integer smaller or equal to \(x\).\(^{29}\)

\(^{29}\)We are also using the convention \(\binom{n}{y} = 0\) if \(y > n\).
The problem is analogous for a minority voter. The relevant equations are:

\[
\tilde{\pi}_m^T = I_Q [(g/G)S_m] \left\{ \sum_{S_m=0}^{M} \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \right\}
\]

\[
\left( \frac{m-1}{(g/G)S_m} \right) F(c_m)^{(g/G)S_m} [1 - F(c_m)]^{m-1 - (g/G)S_m} \right\}
\]

\[
\tilde{\pi}_m^{T-1} = I_Q [(g/G)S_m] \sum_{S_m=1}^{M} \left\{ \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \right\}
\]

\[
\left( m-1 \right) (g/G)S_m - 1 \right) F(c_m)^{(g/G)S_m-1} [1 - F(c_m)]^{m-1 - [(g/G)S_m - 1]} \right\}
\]

\[
\tilde{\pi}_m^W = (1 - I_Q [(g/G)S_m]) \sum_{S_m=0}^{M} \left\{ \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \right\}
\]

\[
\left( \frac{m-1}{[(g/G)S_m]} \right) F(c_m)^{(g/G)S_m} [1 - F(c_m)]^{m-1 - [(g/G)S_m]} \right\}
\]

The probabilities of the minority winning different numbers of position can be derived as under MP, but taking into account that the number of candidates, in each party, now may differ from the number of seats. The probability of the minority losing all seats must be 0 if \( G < K \); if instead \( G \geq K \), then as before it equals the probability that either all minority candidates receive strictly lower votes than the majority candidates, or that all candidates are tied but minority candidates lose all tie-breaks. That is:

\[
\text{Pr}(W_m = 0 | G \geq K) =
\]

\[
= \sum_{S_m=1}^{M} \binom{M}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \sum_{S_m=0}^{X(S_m)} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{m - S_m} +
\]

\[
+ \sum_{S_m=0}^{M} \binom{M}{S_m} \left( \frac{m}{(g/G)S_m} \right) F(c_m)^{S_m} [1 - F(c_m)]^{M - S_m} \times
\]

\[
\times F(c_m)^{(g/G)S_m} [1 - F(c_m)]^{m - (g/G)S_m} I_Q [(g/G)S_m] \left( \frac{G}{K} \right) / \left( \binom{G + g}{K} \right)
\]
where:

\[ X(S^M) = \begin{cases} (g/G)S^M - 1 & \text{if } (g/G)S^M \text{ is an integer} \\ \lfloor (g/G)S^M \rfloor & \text{otherwise} \end{cases} \]

The probability of electing \( w \) minority candidates, with \( w \in (0, g) \) is 0 if \( K - G > w \); it equals the probability that all candidates are tied and \( m \) wins \( w \) tie-breaks if \( K - G < w \), and equals the probability either that all are tied and \( m \) loses all tie-breaks or that all \( m \) candidates receive fewer votes if \( K - G = w \). Thus:

\[
\Pr(W_m = w|K - G \leq w) = \\
= \sum_{S^M=0}^{M} \binom{m}{(g/G)S^M} F(c_m)^{(g/G)S^M} [1 - F(c_m)]^{m-(g/G)S^M} \times \\
\times \binom{M}{S^M} F(c_m)^{S^M} [1 - F(c_m)]^{M-S^M} I_Q[(g/G)S^M] \left( \frac{g}{w} \right) \left( \frac{G}{K - w} \right) \left( \frac{G + g}{K} \right) + \\
+ I_{K-G=w} \sum_{S^M=1}^{M} \binom{M}{S^M} F(c_m)^{S^M} [1 - F(c_m)]^{M-S^M} \times \\
\left( \sum_{M=0}^{S^M} \binom{m}{S^S} F(c_m)^{S^S} [1 - F(c_m)]^{m-S^m} \right)
\]

where \( I_Q[(g/G)S^M] \) and \( X(S^M) \) are defined as above, and \( I_{K-G=w} \) is an indicator function taking value 1 if \( K - G = w \) and 0 otherwise.

Finally, the probability of electing \( g \) minority candidates equals 1 if \( K - G \geq g \), it equals the probability that either all minority candidates receive more votes or that all candidates are tied and the \( g \) minority candidates win all tie-breaks.
That is:

\[
\Pr(W_m = g | K - G < g) = \sum_{S_m=0}^{M} \binom{m}{(g/G)S_m} F(c_m)^{(g/G)S_m} [1 - F(c_m)]^{m-(g/G)S_m} \times \\
\times \binom{M}{S_M} F(c_M)^{S_M} [1 - F(c_M)]^{M-S_M} \sum_{S_m=0}^{M} \binom{M}{S_M} F(c_M)^{S_M} [1 - F(c_M)]^{M-S_M} \times \\
\left( \frac{G}{K} \right)^{G - (g/G)S_m} / \left( \frac{G + g}{K} \right) + \\
\sum_{S_m=1}^{m} \binom{m}{S_m} F(c_m)^{S_m} [1 - F(c_m)]^{m-S_m} \times \\
\left( Y(S_m) \right) \sum_{S_M=0}^{M} \binom{M}{S_M} F(c_M)^{S_M} [1 - F(c_M)]^{M-S_M}
\]

where:

\[
Y(S_m) = \begin{cases} 
(G/g)S_m - 1 & \text{if } (G/g)S_m \text{ is an integer} \\
\lfloor (G/g)S_m \rfloor & \text{otherwise}
\end{cases}
\]
8 References


Report and Minutes of Evidence of the Select Committee on School Board Voting, P.P., (1884/85), 78.


