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A FUNCTIONAL APPROACH TO REVEALED PREFERENCE

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Abstract. We develop a systematic, functional approach to revealed preference tests based on completing preferences. Our approach is based on the notion of sequential closure, which generalizes the notion of transitive closure. We show that revealed preference tests developed for various decision theories can be seen as special cases of our approach. We also illustrate the approach constructing revealed preference tests for theories of decision under uncertainty whose revealed preference implications had not been studied before.

1 Introduction

The revealed preference approach to consumer choice, pioneered by Samuelson (1938), builds on the fact that, although we cannot observe the complete preference relation profiles of economic agents, we can observe their choices over some budget sets. Starting with the work Richter (1966) and Afriat (1967), this approach has been used to construct tests of rational decision making (see Chambers and Echenique, 2016, for a recent comprehensive overview).

Currently there are two complementary approaches to revealed preference theory. The first approach was pioneered by Afriat (1967), and

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further developed by Diewert (1973) and Varian (1983), among others. This approach relies on the concavity of utility functions and the linearity of budget constraints. Given that the consumer problem is a concave optimization problem with linear constraints, the slope of the supergradient of the utility function is known and is given by relative prices. Consistency of the data with basic consumer demand theory (i.e. with the properties of transitivity and monotonicity) can be tested using the Afriat inequalities (also known as supergradient inequalities) obtained from the linear programing problem associated to the consumer decision. Proceeding similarly to Afriat (1967), it is possible to show that the system of Afriat inequalities is necessary and sufficient for consistency of the data with the theory, i.e. there are no additional revealed preference implications. Although this approach is useful and powerful, it relies as explained on the concavity of the utility function and the linearity of budget sets.

An alternative approach follows the seminal work of Richter (1966), and seeks to find ways to complete the revealed preference relation in such a way as to preserve transitivity and monotonicity, as well as other desired properties. This approach has been used variously to provide revealed preference conditions for relaxations of transitivity (Duggan, 1999), for upper semi-continuous utility representation (Bossert et al., 2002), for revealed voting by committees (Gomberg, 2018) and for continuous utility representation on general topological spaces (Nishimura et al., 2017).[1] The scope of this approach extends well beyond the classical consumer problem and is potentially much more general than Afriat's. There has been, however, no common methodology to construct revealed preference tests from this perspective.

This paper provides a generalized approach to construct revealed preference tests by completing preference relations using the notion of closure. Recall that the transitive closure can be defined as a function that adds a pair (x, y) (reads as "x is preferred to y") to the preference relation if there is a sequence $z_1, \ldots, z_n \in X$, with $x = z_1, y = z_n$, such that each pair (z_i, z_{i+1}) is already in the preference relation. More generally, we can think of a closure that adds a pair (x, y) to the preference

[1] Although Nishimura et al. (2017) does not explicitly refer to preference extensions, it relies on the Levin (1983) extension theorem, which is a classical result on completion of preference relations.

relation if there is a sequence $z_1, \ldots, z_n \in X$ with $z_1 = x$ and $z_n = y$, and of functions $f_1, \ldots, f_{n-1}$ from a given family of functions such that each $f_j(z_j, z_{j+1})$ is already in the preference relation. By choosing judiciously the members of the family of functions, one can impose additional properties, besides transitivity. For instance, the family of functions $f(u, v) = (\alpha u, \alpha v)$ for some $\alpha \in \Re_{++}$ imposes homotheticity. We refer to such object as a sequential closure and show that if a decision theory can be represented by a sequential closure, a revealed preference condition holds, generalizing the notions of *Richter congruence* and *Suzumura consistency*. Moreover, we show that sequential closures satisfy the conditions of Demuynck (2009) extension theorem, and therefore a revealed preference test can be easily constructed for each of them. These tests can be represented as finite linear programming problems using Motzkin (1951) transposition theorem.

We provide examples of decision theories which can be represented by sequential closures, including properties such as transitivity, completeness, homotheticity, quasilinearity of preferences, independence (expected utility), gambling independence (Diecidue et al., 2004), and range-dependent independence (Kontek and Lewandowski, 2017). In each of these cases, we present computationally efficient implementations of the corresponding revealed preference tests.

The remainder of this paper is organized as follows. Section 2 presents necessary definitions and introduces formally the notion of sequential closure. Section 3 presents the generalized extension theorem for sequential closures and shows that the above mentioned theories can be represented as sequential closures. Section 4 revisits revealed preference theory, and provides computationally efficient implementations of revealed preference tests. Section 5 provides some concluding remarks. All proofs omitted can be found in the Appendix.

2 Preliminaries

We need to introduce three components of our approach before stating the main result: (i) preference relations and extensions of preference relations, (ii) functions over preference relations–the tool we use to extend the original (incomplete) preference relation, and (iii) some properties of functions over preference relations.

2.1 Preference Relations. Let X be the space of alternatives. A set $R \subseteq X \times X$ is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$. We denote the inverse relation $R^{-1} = \{(x, y)|(y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \setminus I(R)$. We denote the incomparable part by $N(R) = X \times X \setminus (R \cup R^{-1})$.

We list below the standard properties of a preference relation representing rational choice:

Definition 1. *A preference relation R satisfies:*

- ***completeness*** *if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \emptyset$).*
- ***transitivity*** *if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.*

A driving idea in this paper is to extend incomplete preference relations including additional comparisons of pairs of alternatives while preserving the asymmetric part of the original preference relation:

Definition 2. *A preference relation R' is an* ***extension*** *of R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

Next we introduce an equivalent definition of extension.

Lemma 1. *Let $R \subseteq R'$. $R \preceq R'$ if and only if $P^{-1}(R) \cap R' = \emptyset$.*

2.2 Functions over preference relations. In this section we consider general functions $F : \mathcal{R} \to \mathcal{R}$ defined over the set of preference relations which may be used to extend an incomplete preference relation. The simplest example of such function is the *transitive closure*, which adds (x, z) to R for each $(x, y) \in R$ and $(y, z) \in R$. Being more precise, $(x, y) \in T(R)$ if and only if there is a sequence of elements $S = s_1, \ldots, s_n$, such that for every $j = 1, \ldots, n-1$ we have $(s_j, s_{j+1}) \in R$, where T stays for transitive closure.

We generalize below the notion of transitive closure in an intuitive way that is useful in order to induce particular properties. Let $f : Y \to Z$ be a function, where $Y, Z \subseteq X$. A function $f^{-1} : Z \to Y$ is an **inverse** of f if $f^{-1}(f(x)) = x$. A function f that has an inverse is said to be **invertible**. Denote by $f \circ f'$ a **superposition** of functions f and f', that is $[f \circ f'](x) = f(f'(x))$. Further we refer to $f(x, y) = (f(x), f(y))$ unless we need directly the property that the function is similar for both (x, y). This abuse of notation is done to make the further exposition simpler. Denote the **identify mapping** by $I(x, y) = (x, y)$.

Definition 3. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be a* **sequential closure** *if $(x, y) \in F(R)$ if and only if there are finite sequences $s_1, \dots, s_n \in X$ with $x = z_1, y = z_n$, and $f_1, \dots, f_{n-1} \in \mathcal{F}$, such that $f_j(z_j, z_{j+1}) \in R$ for $j = 1, \dots, n$, where $\mathcal{F}$ is a family (collection) of invertible functions $f : Y \to Z$, where $Y, Z \subseteq X^2$, that contains the identity mapping and is closed under taking inverse ($f \in \mathcal{F}$ implies $f^{-1} \in \mathcal{F}$) and closed under taking the superposition ($f, f' \in \mathcal{F}$ implies $[f \circ f'] \in \mathcal{F}$).*

Note that the transitive closure is a sequential closure with associated family of functions being a singleton with the identity mapping as unique element: $\mathcal{F} = \{I\}$. Hence, the transitive closure is the minimal possible sequential closure.

2.3 Functions and preference relations. It is well known that a preference relation is extendable by a transitive closure if and only if it satisfies *Suzumura consistency* (see e.g. Bossert, 2018). That is, the transitive closure of the preference relation does not contain strict preference cycles. The following definition extends this notion of consistency to general functions over preference relations.

Definition 4. *A preference relation is said to be* **externally consistent** *with a function $F : \mathcal{R} \to \mathcal{R}$ if $P^{-1}(R) \cap F(R) = \emptyset$.*

A preference relation R^* is said to be a **fixed point** of F if $F(R^*) = R^*$. If the closure imposes some particular properties, then every fixed point satisfies the corresponding properties. Note that every fixed point of the transitive closure is a transitive preference relation.

3 Results

Theorem 1. *Let $F : \mathcal{R} \to \mathcal{R}$ be a sequential closure. There is a complete, fixed point extension of R if and only if R is externally consistent.*

Theorem 1 provides a criterion for the existence of a preference extension satisfying the properties imposed by any particular sequential closure. Next, we provide several examples of sequential closures. Each of them is designed in order to impose a particular set of properties, and therefore can be used in order to construct a revealed preference tests of those properties.

3.1 Examples.

3.1.1 Transitive Closure. Denote the **transitive closure** by

$$T : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, \ldots, s_n$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \ldots, n-1$, and $s_1 = x$ and $s_n = y$. In this case we have $\mathcal{F} = \{I\}$. Note that that identity mapping is an inverse to itself, and the composite of the identity mapping with itself is the identity. Therefore, $\mathcal{F}$ is a collection of invertible functions closed with respect to taking superposition and inversion. Moreover, R is a transitive preference relation if and only if it is fixed point of T (see Demuynck (2009) for the proof).

3.1.2 Homothetic Closure. A preference relation R satisfies **homotheticity** if $(x, y) \in R$ if and only if $(ax, ay) \in R$ for all $a \in \mathbb{R}_{++}$ and $x, y \in X$.

Denote the **homothetic closure** by

$$H : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in H(R)$ if and only if there is a finite sequence $s_1, \ldots, s_n \in X$ with $s_1 = x, s_n = y$ such that there is some $\alpha_j \in \mathbb{R}_{++}$ such that $(\alpha_j s_j, \alpha_j s_{j+1}) \in R$ for every $j = 1, \ldots, n-1$. In this case

$$\mathcal{F} = \{f_\alpha : X^2 \to X^2 | f_\alpha(x, y) = (\alpha x, \alpha y) \text{ for some } \alpha \in \mathbb{R}_{++}\}.$$

Note that every function f_α is invertible and its inverse $f_{\frac{1}{\alpha}}$ is also in $\mathcal{F}$. Moreover, the superposition of functions f_{α_j} can be represented as $f_{\prod_j \alpha_j}$, which is also in $\mathcal{F}$. Therefore, $\mathcal{F}$ is a collection of invertible functions closed with respect to taking superposition and inversion. Moreover, R is a homothetic and transitive preference relation if and only if it is a fixed point of H (see Demuynck (2009) for the proof).

3.1.3 Quasilinear Closure. A preference relation R satisfies **quasilinearity** if $(x,y) \in R$ if and only if $(x+ae_i, y+ae_i) \in R$ for all $a \in \mathbb{R}$ and $x,y \in X$, where e_i is a vector with zeros everywhere and 1 at the i-th place.

Denote the **quasilinear closure** by

$$Q : \mathcal{R} \to \mathcal{R},$$

where $(x,y) \in Q(R)$ if and only if there is a finite sequence $s_1,\ldots,s_n$ such that there is $\alpha_j \in \mathbb{R}$ such that $(s_j+\alpha_j e_i, s_{j+1}+\alpha_j e_i) \in R$ for every $j=1,\ldots,n-1$, and $s_1 = x$ and $s_n = y$. In this case

$$\mathcal{F} = \{f_\alpha : X^2 \to X^2 | f_\alpha(x,y) = (x+\alpha e_i, y+\alpha e_i) \text{ for some } \alpha \in \mathbb{R}\}.$$

Note that every function f_α is invertible and its inverse $f_{-\alpha}$ also belongs to $\mathcal{F}$. Moreover, every superposition of functions f_{α_j} is $f_{\sum_j \alpha_j}$ that also belongs to $\mathcal{F}$. Therefore, $\mathcal{F}$ is a collection of invertible functions closed with respect to taking superposition and inversion. Moreover, R is a quasilinear and transitive preference relation if and only if it is fixed point of Q (see Castillo and Freer (2016) for the proof).

3.1.4 Independent Closure. We define the space of alternatives now as $\triangle(X)$ given some underlying outcome space X, with the interpretation that the objects of choice are lotteries over the set X. As customary, for $x,y \in \triangle(X)$ we write $\lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$ to indicate the (composite) lottery in which the lottery x occurs with probability λ and the lottery y with probability $1-\lambda$. A preference relation $R \subseteq \triangle(X) \times \triangle(X)$ satisfies **independence** if for every $\lambda \in (0,1)$ and $x,y,z \in \triangle(X)$,

$$(x,y) \in R \iff (\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R.$$

Denote the **transitive and independent closure** by

$$TI : \mathcal{R} \to \mathcal{R},$$

where $(x,y) \in TI(R)$ if and only if there is a finite sequence $x = s_1,\ldots,s_n = y$ such that there are $\lambda_j \in (0,1]$ and $z_j \in \triangle(X)$ such that $(\lambda_j s_j + (1-\lambda_j)z_j, \lambda_j s_{j+1} + (1-\lambda_j)z_j) \in R$, where $s_j, s_{j+1}, z_j \in \triangle(X)$. The previous statement implies that $\lambda_j s_j + (1-\lambda_j)z_j, \lambda_j s_{j+1} + (1-$

$\lambda_j)z_j \in \triangle(X)$. In this case we have

$$\mathcal{F} = \{f_\alpha : f_{\lambda,z}(x,y) = (\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \text{ such that} \\ x, y, \lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in \triangle(X)\}$$

for some $\lambda > 0$ and $z \in \triangle(X)$. Note that if $\lambda > 1$, we need to restrict the domain of the function so that the range is contained in $\triangle(X)^2$. With this proviso, for every $f_{\lambda,z}(x,y)$ there is an inverse $f_{\frac{1}{\lambda},z}(x,y)$ which is also in $\mathcal{F}$. Therefore, $\mathcal{F}$ is a collection of invertible functions closed with respect to inversion. Moreover, $\mathcal{F}$ is closed under taking superposition. To show this, take any two functions $f_{\lambda,z}$ and $f_{\lambda',z'}$. Their superposition is $f_{\lambda,z} \cdot f_{\lambda',z'} = f_{\tilde{\lambda},\tilde{z}}$, where $\tilde{\lambda} = \lambda\lambda'$ and $\tilde{z} = \frac{\lambda'(1-\lambda)}{1-\lambda'\lambda}z + \frac{1-\lambda'}{1-\lambda'\lambda}z'$ if $\lambda'\lambda \neq 1$ and an arbitrary element of $\triangle(X)$ otherwise, so that $f_{\lambda,z} \cdot f_{\lambda',z'} \in \mathcal{F}$. Moreover, R is a independent and transitive preference relation if and only if it is fixed point of TI (see Demuynck and Lauwers (2009) for the proof).

3.1.5 Gambling Independent Closure. A preference relation R satisfies **gambling independence** if for every $\lambda \in (0,1)$ and $x, y, z \in \triangle(X) \setminus X$,

$$(x,y) \in R \iff (\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R.$$

Gambling independence allows agent to get some intrinsic utility or disutility from "gambling" – i.e. choosing a non-degenerate lottery instead of a certain outcome. The independence axiom imposes constraints on the comparisons between non-generate lotteries and not in their comparisons with certain outcomes.

Denote the **transitive and gambling independence closure** by

$$TGI : \mathcal{R} \to \mathcal{R},$$

where $(x,y) \in TGI(R)$ if and only if there is a sequence $x = s_1, \ldots, s_n = y$ such that

- $(s_j, s_{j+1}) \in R$, or
- $s_j, s_{j+1} \in \triangle(X) \setminus X$ and there are $z \in \triangle(X) \setminus X$ and $\lambda > 0$ such that $(\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z) \in R$ and $\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z \in \triangle(X) \setminus X$.

In this case we have two type of functions: the identity function as in the transitive closure, and mixtures in the same fashion as for the transitive and independent closure, but defined only over risky choices, i.e.

excluding the for sure alternatives. Note that in this case every function $f_{\lambda,z} \in \mathcal{F}$ is defined by $\lambda > 0$ and $z \in \triangle(X) \setminus X$, and $f_{\lambda,z}(x,y) = (\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)z)$ such that $\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)z \in \triangle(X)$. Note that for every $f_{\lambda,z}(x,y)$ there is an inverse $f_{\frac{1}{\lambda},z}(x,y)$ which is also in $\mathcal{F}$. Therefore, $\mathcal{F}$ is a collection of invertible functions closed with respect to inversion. Moreover, the $\mathcal{F}$ is closed under taking superposition of the functions by the same token as the corresponding family of functions for the TI, with the only difference that we consider only the pure lotteries instead of the entire set of elements of the simplex. Moreover, R is a gambling independent and transitive preference relation if and only if it is a fixed point of TGI (see the Appendix for the proof).

3.1.6 Within-Range Independent Closure. The idea behind range-dependent utility is that there is a possibly different Bernoulli utility function for each interval $[x_l, x_u]$ taken from a collection of exogenously given intervals in $X \subseteq \Re$. Denote the set of all (allowed) closed intervals by $\mathcal{I}$. A preference relation R satisfies **within-range independence** if for every $[x_l, x_u] \in \mathcal{I}$, $\lambda \in (0,1)$, and $x, y, \lambda x+(1-\lambda)z, \lambda y+(1-\lambda)z \in \triangle([x_l, x_u])$,

$$(x,y) \in R \iff (\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R.$$

As the name makes it clear, the property of within-range independence imposes independence within every allowed range in $\mathcal{I}$.

Denote the **transitive and within-range independence closure** by

$$TWI : \mathcal{R} \to \mathcal{R},$$

where $(x,y) \in TWI(R)$ if and only if there is a sequence $x = s_1, \ldots, s_n = y$ such that

- $(s_j, s_{j+1}) \in R$, or
- there are $[x_l, x_u] \in \mathcal{I}$, $z \in \triangle(X)$, and $\lambda > 0$ such that $s_j, s_{j+1}, \lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z \in \triangle([x_l, x_u])$ and $(\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z) \in R$.

Note that since every set of lotteries with outcomes being in the particular interval is closed under taking a convex combination, then the TWI can be perceived as a multiple TIs, one for every interval in $\mathcal{I}$. Therefore, a preference relation is fixed point of TWI if and only if it is transitive and satisfies within-range independence. Moreover, we can construct a family of functions that is closed under taking the inverse and taking the superposition by exactly the same token as for TI.

4 Revealed Preferences Revisited

Let $\mathcal{B}$ be a collection of budgets where every $B \in \mathcal{B}$ is a subset of X, and let $C : \mathcal{B} \to 2^X$ be the observed choice function. Then $(C, \mathcal{B})$ specifies the consumption experiment. An experiment is said to be finite if $\mathcal{B}$ is a finite collection of budgets. Moreover, assume that X is endowed with a partial order $\geq$, denoting its strict part by $>$. Rationalization includes strict monotonicity (in our terms it is expressed as $\geq \preceq R$); one can easily check that otherwise every finite experiment can be rationalized. We say that $x \in \max(B, R)$ if $x \in B$ and there is no $y \in B$ such that $(y, x) \in R$.[2] For any budget B, we denote

$$B^{\downarrow} = \{x \in B : \text{ there is } y \in B/\{x\} \text{ such that } y \geq x\}$$

and

$$B^{\downarrow\downarrow} = \{x \in B : \text{ there is } y \in B \text{ such that } y > x\}.$$

Given a sequential closure F, a consumption experiment is said to be **rationalizable** if there is a complete and monotone preference relation $\geq \preceq R^*$ that is a fixed point of F such that

$$C(B) \subseteq \max(B, R^*) \text{ for every } B \in \mathcal{B}.$$

We say that a sequential closure F is idempotent if $F(F(R)) = F(R)$. Denote by R_E a **revealed preference relation**, constructed as $(x, y) \in R_E$ if there is $B^t \in \mathcal{B}$ such that $x \in C(B^t)$ and $y \in B^t$.

Corollary 1. *Let F be an idempotent sequential closure. An experiment $(\mathcal{B}, C)$ is rationalizable if and only $>^{-1} \cap F(R_E \cup \geq) = \emptyset$.*

The condition in Corollary 1 is a generalization of the monotone congruence axiom in Nishimura et al. (2017). Further we present some examples of how the criterion from Corollary 1 generalizes existing revealed preference axioms. We present computationally efficient tests for the external consistency condition for the closures used above. We

[2] If the relation is complete equivalence, the definition would guarantee that $(x, y) \in R$ for all $y \in B$.

also show that utility representations can be obtained for experiments satisfying the external consistency condition under some additional assumptions.

4.1 Transitive Rationalization.

Definition 5. *A consumption experiment* $E = \{(x_i, B_i\}_{i=1}^n$ *satisfies* ***GARP*** *if for any sequence* $x_{k_1} \in B^{\downarrow}_{k_2}, \ \ldots, \ x_{k_{n-1}} \in B^{\downarrow}_{k_n}$ *we have* $x_{k_n} \notin \mathrm{int}(B^{\downarrow}_{k_1})$.

GARP is equivalent to the monotone congruence condition, which delivers the following corollary.

Corollary 2. *An experiment is rationalizable with respect to* T *if and only if it satisfies GARP.*

Moreover, using a result from Nishimura et al. (2017) we can guarantee that a finite experiment is rationalizable with a continuous utility representation.

4.2 Homothetic Rationalization.

Definition 6. *A consumption experiment* $E = \{(x_i, B_i\}_{i=1}^n$ *satisfies* ***HARP*** *if for any sequence of distinct elements* $x_{k_1}, ..., x_{k_n}$ *and*

$$(\beta_2, \beta_2, \ldots, \beta_n) \in \mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$$

such that $x_{k_{j+1}}/\beta_{j+1} \in B^{\downarrow}_{k_j}$ *for* $j = 2, \ldots, n$*, we have*

$$x_{k_1} \times \prod_{j=2}^{n} \beta_j \notin \mathrm{int}(B^{\downarrow}_{k_n}).$$

HARP is tractable because we do not have to check the condition for sequences involving every possible β, but checking only for minimal ones is sufficient. For finite experiments, the number of β that needs to be checked is finite, and therefore the condition can be checked in the finite time.

Corollary 3. *An experiment is rationalizable with respect to* H *if and only if it satisfies HARP.*

Using a result from Forges and Minelli (2009) we can get utility rationalization for budgets which are not necessarily linear as long as the experiment is finite. This result itself delivers a significant generalization of Varian (1983) result.

4.3 Quasilinear Rationalization.

Definition 7. *A revealed preference relation satisfies **QARP** with respect to the i-th component if for any sequence of distinct elements* $x_{k_1}, ..., x_{k_n}$ *and* $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$*, such that* $x_{k_2} - \alpha e_i \in B^{\downarrow}_{k_1}$ *and* $x_{k_{j+1}} - \beta_{j+1} e_i \in B^{\downarrow}_{k_j}$ *for* $j = 2, \ldots, n-1$*, then* $x_{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i \notin \operatorname{int}(B^{\downarrow}_{k_n})$.

QARP is equivalent to cyclical monotonicity conditions which are easily testable.

Corollary 4. *An experiment is rationalizable with respect to Q if and only if it satisfies QARP.*

Using a result from Castillo and Freer (2016) we can obtain a continuous utility representation of an experiment that satisfies QARP as long as the experiment to be finite.

4.4 Expected Utility Rationalization. Next we construct a test for TI-consistency of finite experiments. We assume that lotteries chosen can be observed, and that the number of certain outcomes is finite. This test and the following are expressed as linear programs. We introduce some additional notation for that purpose. Given a lottery L denote by $\operatorname{supp}(L)$ the support of it – the set of certain outcomes which happen with non-zero probability. Denote by $L^t = C(B^t)$ the lottery chosen from budget $B^t \in \mathcal{B}$. Denote by $\pi_L(x)$ the probability of outcome x under lottery L. Denote by

$$\mathcal{X} = \bigcup_t \operatorname{supp}(L^t) \cup \{0\}$$

the set of all outcomes that occur with positive probability. Finally let

$$\mathcal{L} = \{L \in \triangle(X) :\ \operatorname{supp}(L) \subseteq \mathcal{X}\}.$$

Expected utility (with a monotone Bernoulli utility function) implies that the following conditions should be satisfied:

$$\sum_{x \in \mathcal{X}} \pi_{L^t}(x) u(x) \geq \sum_{x \in \mathcal{X}} \pi_L(x) u(x)\ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow},$$
$$\sum_{x \in \mathcal{X}} \pi_{L^t}(x) u(x) > \sum_{x \in \mathcal{X}} \pi_L(x) u(x)\ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow\downarrow}.$$

Consider a matrix A^R with $|\mathcal{X}|$ columns. Each row corresponds to one of the weak inequalities described above. Take a row corresponding to $L^t - L$ such that L^t is a chosen lottery and $L \in \mathcal{L} \cap (B^t)^{\downarrow}$. Hence, in each cell, determined by the column corresponding to $x \in \mathcal{X}$, we plug in $a_{L-L,x} = \pi_{L^t}(x) - \pi_L(x)$. Therefore, the system of inequalities above can be rewritten in a matrix form as follows, with A^R being a collection of rows which correspond to weak inequalities. Similarly, we can construct A^P as a collection of rows which correspond to strict inequalities.

$$\text{(EU)} \qquad \begin{cases} A^R u \geq 0, \\ A^P u > 0. \end{cases}$$

Corollary 5. *An experiment is rationalizable with respect to TI if and only if it there is $u : \mathcal{X} \to \mathbb{R}$ such that the system EU has a solution.*

Polisson et al. (2017) shows that existence of a solution to this system of inequalities is equivalent to the existence of expected utility representation. Hence, Corollary 5 implies expected utility rationalization as well.

4.5 Gambling Independence Utility Rationalization. Diecidue et al. (2004) introduces a utility of gambling by defining separately an elementary utility function $u : X \to \mathbb{R}$ used for expected utility comparisons between non-degenerate lotteries and another elementary utility $v : X \to \mathbb{R}$ used for comparisons between certain outcomes. In order to accommodate the utility of gambling we need to modify the inequalities for expected utility taking into account for the existence of both utility functions. We have

$$\begin{aligned}
&\sum_{x\in\mathcal{X}} \pi_{L^t}(x)u(x) \geq \sum_{x\in\mathcal{X}} \pi_L(x)u(x) \ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow}, \\
&\sum_{x\in\mathcal{X}} \pi_{L^t}(x)u(x) \geq v(x) \ \forall L \in \mathcal{X} \cap (B^t)^{\downarrow}, \\
&v(x^t) \geq \sum_{x\in\mathcal{X}} \pi_L(x)u(x) \ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow}, \\
&v(x^t) \geq v(x) \ \forall L \in \mathcal{X} \cap (B^t)^{\downarrow}.
\end{aligned}$$

We proceed similarly with respect to strict inequalities. As in the previous case we can write a matrix B with $2|\mathcal{X}|$ columns, $|\mathcal{X}|$ columns for the outcomes with respect to non-degenerate lotteries and $|\mathcal{X}|$ columns for certain outcomes. We can use B rewrite the inequalities above in the matrix form:

$$\begin{cases} B^R u \geq 0, \\ B^P u > 0. \end{cases}$$

Next, we apply Motzkin (1951) transposition theorem to obtain an alternative system, which should have no solution.

$$\begin{cases} q^R B^R + q^P B^P \geq 0, \\ q^R \geq 0, \\ q^P > 0. \end{cases}$$

Furthermore, we cannot apply the convex combination to the pairs which contain at least one certain outcome. Denote by q_L the part of the vector which corresponds to the lotteries and by q_X the part of the vector which corresponds to the rows with certain outcome included. Hence, we further restrict the system to the following.

$$\text{(GU)} \qquad \begin{cases} q^R B^R + q^P B^P \geq 0, \\ q^R \geq 0, \\ q^P > 0, \\ q_X \in \{0,1\}^{|q_X|}. \end{cases}$$

Therefore, we obtain a mixed integer linear program, which can be efficiently solved. Moreover, the number of integer variables is limited by the possible number of certain outcomes in the budget intersection with the lotteries over the lattice.

Corollary 6. *An experiment is rationalizable with respect to TI if and only if it there is no q such that the system GU has a solution.*

4.6 Range-Dependent Rationalization. For the range-dependent utility rationalization there is a Bernoulli utility function for every interval $u : [x_l, x_u] \to \mathbb{R}$ with $[x_l, x_u] = I \in \mathcal{I}$. However, in this case there are multiple utility levels for the same lottery. Hence, for every interval the expected utility inequalities should be satisfied.

$$\sum_{x\in\mathcal{X}} \pi_{L^t}(x)u^I(x) \geq \sum_{x\in\mathcal{X}} \pi_L(x)u^I(x) \ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow} \cap \triangle(I),$$
$$\sum_{x\in\mathcal{X}} \pi_{L^t}(x)u^I(x) > \sum_{x\in\mathcal{X}} \pi_L(x)u^I(x) \ \forall L \in \mathcal{L} \cap (B^t)^{\downarrow\downarrow} \cap \triangle(I).$$

The inequalities can be rewritten in the matrix form given the matrix A with $|\mathcal{X}|$ columns and rows corresponding to different lotteries.

$$\begin{cases} A_I^R u^I \geq 0, \\ A_I^P u^I > 0. \end{cases}$$

Note that the matrices can be different for different intervals, because matrix stands for the revealed preference relations over the interval. However, we also need to impose the transitivity. For this purpose let us first use the Motzkin (1951) transposition theorem to obtain the alternative to the inequalities.

$$\begin{cases} q_I^R A_I^R + q_I^P A_I^P = 0, \\ q_I^R \geq 0, \\ q_I^P > 0. \end{cases}$$

This is an alternative system, and therefore, it should have no solution. Otherwise there is a violation of within-range independence. Furthermore, we may have the violation of transitivity coming from different intervals, and therefore, we need to control that independence on one interval does not impose the preference reversal to the independence on another interval.

$$\begin{cases} \sum_{i\in\mathcal{I}} \left(q_I^R A_I^R + q_I^P A_I^P\right) = 0, \\ q^R \geq 0, \\ q^P > 0. \end{cases}$$

We need to take care of pairs of lotteries which do not belong jointly to any of the intervals. They can affect the relation via the transitivity over the space of lotteries. Denote by $A_{\bar{I}}$ the corresponding matrix and by $q_{\bar{I}}$ the corresponding vector, which would be binary, because we can only use the transitive closure for this case.

$$\text{(RDU)} \qquad \begin{cases} \sum_{i\in\mathcal{I}} \left(q_I^R A_I^R + q_I^P A_I^P\right) + (q_{\bar{I}}^R A_{\bar{I}}^R + q_{\bar{I}}^P A_{\bar{I}}^P) = 0, \\ q^R \geq 0, \\ q^P > 0, \\ q_{\bar{I}} \in \{0,1\}^{|q_{\bar{I}}|}. \end{cases}$$

For range-dependent utility we also obtain a mixed integer linear program. The number of integer variables in this case depends on the interval structure imposed.

Corollary 7. *An experiment is rationalizable with respect to TI if and only if it there are no q_I^R, q_I^P and δ_I for every $I \in \mathcal{I}$ such that the system RDU has a solution.*

5 Concluding Remarks

The paper provides a comprehensive approach to revealed preferences based on a generalization of transitive closure which we call sequential closure. We refer to this as the functional approach to the revealed preferences. We show that if a decision theory can be represented by a sequential closure, then it can be tested using a simple revealed preference condition (monotone congruence). Moreover, we show that various theories can be represented by sequential closures, including such properties as transitivity, homotheticity, quasilinearity, independence, gambling independence, and within-range independence. Revealed preference tests for the first four theories have been provided before in the literature and we show that the existing tests are the special cases of the congruence axiom we state for the abstract sequential closure. The revealed preference tests provided for the last two theories are to the best of our knowledge new.

We believe the functional approach opens fruitful venues for further research. In particular, the examples provided show that under some additional restrictions on the functions used to construct the closure, revealed preference tests can be expressed as linear programming problems. This suggests that it may be possible to construct a general, abstract test of revealed preference. Another interesting question is the logical restatement of the functional closures and the understanding the empirical content behind this entire class of theories (as in Chambers et al., 2014).

Appendix: Proofs

In order to proceed to the proofs we need to introduce some additional notations and supplementary results.

Definition A.1. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

- ***monotone*** *if for all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,*
- ***closed*** *if for all $R \in \mathcal{R}$, $R \subseteq F(R)$,*
- ***idempotent*** *if for all $R \in \mathcal{R}$, $F(F(R)) = F(R)$,*
- ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*
- ***weakly expansive*** *if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$,*
- ***expansive*** *if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$ and $P(R) = P(R \cup S)$,*
- ***transitivity-inducing*** *if any preference relation satisfying $R = F(R)$ is transitive,*
- ***separability-preserving*** *any countable set Z, $P(F(P(R))) = P(R)$ and $R \in \mathcal{R}_F^Z$, $F(R)$ is Z-separable.*
- ***convergent*** *$F(R) = R$ where implies that $P(F(P(R))) = P(R)$.*

A monotone, closed, idempotent and algebraic function over preference relations is said to be **algebraic closure**. An algebraic closure that is transitivity-inducing, separability preserving, expansive and convergent is said to be **rational closure**.

Proof of Theorem 1.

Lemma A.1 (Demuynck (2009) Representation Theorem)**.** *Let $F : \mathcal{R} \to \mathcal{R}$ be a weakly expansive algebraic closure. There is a complete, fixed point extension of R if and only if R is externally consistent.*

To complete the proof we are left to show, that every idempotent sequential closure is a weakly expansive algebraic closure. The rest follows from Lemma A.1.

Lemma A.2. *If $F : \mathcal{R} \to \mathcal{R}$ is a sequential closure, then it is a weakly expansive algebraic closure.*

Proof. F **is increasing, i.e.** $R \subseteq F(R)$**.**

Recall that $I \in \mathcal{F}$, therefore, for every $(x, y) \in R$ there is a sequence $x = s_1, s_2 = y$ and $f_1 = I$, such that $I(x, y) \in R$, therefore $(x, y) \in F(R)$.

F **is monotone, i.e.** $R \subseteq R'$ **implies** $F(R) \subseteq F(R')$**.**

Take $(x, y) \in F(R)$, then there are sequences $x = s_1, \ldots, s_n$ and $f_1, \ldots, f_{n-1}$ such that $f_j(s_j, s_{j+1}) \in R \subseteq R'$. This implies that every of those elements is in R' and therefore, $(x, y) \in F(R')$.

F **is algebraic.**

Consider a relation R and an element $(x, y) \in F(R)$, then there are sequences $x = s_1, \ldots, s_n$ and $g_1, \ldots, f_{n-1}$ such that $f_j(s_j, s_{j+1}) \in R$. Let $D = \{s_1, \ldots, s_n\}$ and let $R' = R \cap (D \times D)$. Then, $(x, y) \in F(R')$ and R' is finite by definition.

F **is weakly expansive.**

Take $(x, y) \in N(R)$ and let $R' = R \cup \{(x, y)\}$. Assume on the contrary that R' is not externally consistent, i.e. $P^{-1}(R') \cap F(R') \neq \emptyset$. Therefore, there is $(w, z) \in P^{-1}(R') \cap F(R')$.

Case 1: $(w, z) \neq (y, x)$**.** So, there are sequences $w = s_1, \ldots, s_n$ and $g_1, \ldots, g_{n-1}$ such that $g_j(s_j, s_{j+1}) \in R$. Note that this sequence has to contain (x, y), because R is a fixed point of G, i.e. externally consistent. Let us write this sequence out explicitly given that k is the number such that $s_k = x$:

$$f(w, s_2), \ldots, f_{k-1}(s_{k-1}, s_k), f_k(s_k, s_{k+1}), f_{k+1}(s_{k+1}, s_{k+2}),$$
$$\ldots, f_{n-1}(s_{n-1}, z),$$

where $f_k(s_k, s_{k+1}) = (x, y)$. Therefore, we can reorder the sequence taking into the account that $(z, w) \in R$:

$$f_{k+1}(s_k, s_{k+1}), \ldots, f_{n-1}(s_{n-1}, z), I(z, w), f(w, s_2), \ldots, f_{k-1}(s_{k-1}, x).$$

This implies that $(s_k, s_{k+1}) \in R$, however R is a fixed point of G and $\mathcal{F}$ is closed under taking the inverse, therefore, $f^{-1}(s_k, s_{k+1}) = (y, x) \in R$, since $f(f^{-1}(s_k, s_{k+1})) = (s_k, s_{k+1})$.

Case 2: $(w, z) = (y, x)$**.** If the sequence that adds (y, x) contains (x, y) than we can obtain contradiction using the same argument as above, otherwise it implies directly that $(y, x) \in R$ that is also a contradiction.

F **is idempotent.**

Note that since F is increasing we already know that $F(R) \subseteq F(F(R))$, therefore, we are left to show that $F(F(R)) \subseteq F(R)$. Take an element $(x, y) \in F(F(R))$, then there are sequences $x = s_1, \dots, s_n$ and $f_1, \dots, f_{n-1}$ such that $(f_j(s_j), f_j(s_{j+1})) \in F(R)$. For every j such that $(f_j(s_j), f_j(s_{j+1})) \in F(R) \setminus R$, we know that there are sequences $S^j = s^j_1, \dots, s^j_n$, $f^j_1, \dots, f^j_{n-1}$ such that $s^j_1 = f_j(s_j)$, $s^j_n = f_j(s_{j+1})$ and $(f^j_i(s^j_i), f^j_i(s^j_{i+1})) \in R$. Recall that $\mathcal{F}$ contains only invertible functions, so there is f^{-1}_j, then let $\hat{s}^j_i = f^{-1}_j(s^j_i)$ and $\hat{f}^j_i = [f^j_i \circ f_j](x)$. Recall that $\mathcal{F}$ is closed with respect to taking superposition of functions, therefore, $\hat{f}^j_i \in \mathcal{F}$. So, there are sequences $\hat{S}^j = \hat{s}^j_1, \dots, \hat{s}^j_n$ and $\hat{f}^j_1, \dots, \hat{f}^j_{n-1}$ such that $s_j = \hat{s}^j_1$; $s_{j+1} = \hat{s}^j_n$ and $(\hat{f}^j_i(\hat{s}^j_i), \hat{f}^j_i(\hat{s}^j_{i+1})) \in R$. Therefore, we can incorporate this part as a subsequence in the original sequence and obtain the grand sequence which would have only pairs from R. Therefore, $(x, y) \in F(R)$. $\square$

Gambling Independence.

Lemma A.3. *A preference relation R satisfies gambling independence if and only if $(x, y) \in R$ and $z \in \triangle(X) \setminus X$ implies $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$ where*

- *if $\lambda \in (0, 1)$, then $x, y \in \triangle(X) \setminus X$,*
- *if $\lambda > 1$, then $\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z \in \triangle(X) \setminus X$.*

Proof. **(⇒)** Consider $(x, y) \in R$, $z \in \triangle(X) \setminus X$ and λ. If $\lambda \in (0, 1)$, then $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$ is true (see direct implication). If $\lambda > 1$, denote $w = \lambda x + (1-\lambda)z$ and by $q = \lambda y + (1-\lambda)z$. Then, $x = \frac{1}{\lambda}w + (1 - \frac{1}{\lambda})z$ and $y = \frac{1}{\lambda}q + (1 - \frac{1}{\lambda})z$. Note that in this case $\frac{1}{\lambda} < 1$. Therefore, $(w, q) \in R$, that is $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$.

(⇐) Consider $x, y \in \triangle(X) \setminus X$, such that $(x, y) \in R$; $z \in \triangle(X) \setminus X$ and $\lambda \in (0, 1)$. Then, $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$ is true (see first implication). If there is $x, y, z \in \triangle(X) \setminus X$ and $\lambda \in (0,1)$ such that $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$. Denote by $w = \lambda x + (1-\lambda)z$ and by $q = \lambda y + (1-\lambda)z$. Then, $x = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$ and $y = \frac{1}{\lambda}q + (1-\frac{1}{\lambda})z$. Note that in this case $x, y \in \triangle(X) \setminus X$ since $\frac{1}{\lambda} > 0$. Hence, $(x, y) \in R$ (see second implication). □

Further we will be using the operationalizable definition of gambling independence, since it is closer to the properties induced by $TGI : \mathcal{R} \to \mathcal{R}$.

Lemma A.4. *$R = TGI(R)$ if and only if R satisfies transitivity and gambling independence.*

Proof. **(⇒)** The fact that every fixed point of $TGI : \mathcal{R} \to \mathcal{R}$ is transitive is quite obvious since $T(R) \subseteq TGI(R)$ for every $R \in \mathcal{R}$. Let $x, y, z \in \triangle(X) \setminus X$ such that $(x, y) \in R$. If $\lambda \in (0;1)$, denote by $w = \lambda x + (1-\lambda)z$ and by $q = \lambda y + (1-\lambda)z$. Then, $x = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$ and $y = \frac{1}{\lambda}q + (1-\frac{1}{\lambda})z$. In this case $\frac{1}{\lambda} > 1$ and $x, y, z \in \triangle(X) \setminus X$. Therefore, $(w, q) \in TGI(R) = R$, that is $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$. If $\lambda > 1$ denote by $w = \lambda x + (1-\lambda)z$ and by $q = \lambda y + (1-\lambda)z$. Then, $x = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$ and $y = \frac{1}{\lambda}q + (1-\frac{1}{\lambda})z$. In this case $\frac{1}{\lambda} \in (0;1)$ and $w, q, z \in \triangle(X) \setminus X$. Therefore, $(w, q) \in TGI(R) = R$, that is $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \in R$.

(⇐) Note that we already know that $R \subseteq TGI(R)$, since $TGI : \mathcal{R} \to \mathcal{R}$ is a weakly sequential function. So, we left to show that $TGI(R) \subseteq R$. Further we separate cases, which are different for the depending on whether $\lambda \in (0,1)$ or $\lambda > 1$, because they correspond to the different directions of the implications. Take, $(x, y) \in TGI(R)$, then there is a sequence $x = s_1, \ldots, s_n = y$, such that

- $(s_j, s_{j+1}) \in R$,

- or $s_j, s_{j+1} \in \triangle(X) \setminus X$, and there are $z \in \triangle(X) \setminus X$, $\lambda \in (0;1)$ such that $(\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z) \in R$.
  Denote by $w = \lambda s_j + (1-\lambda)z$ and $q = \lambda s_{j+1} + (1-\lambda)z$. Then $s_j = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$ and $s_{j+1} = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$. In this case $\frac{1}{\lambda} > 1$ and $s_j, s_{j+1}, z \in \triangle(X) \setminus X$. Therefore, gambling independence implies implies that $(s_j, s_{j+1}) \in R$.

- there are $z \in \triangle(X) \setminus X$ and $\lambda > 1$ such that $\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z \in \triangle(X) \setminus X$ and $(\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z) \in R$.
  Denote by $w = \lambda s_j + (1-\lambda)z$ and $q = \lambda s_{j+1} + (1-\lambda)z$. Then $s_j = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$ and $s_{j+1} = \frac{1}{\lambda}w + (1-\frac{1}{\lambda})z$. In this case $\frac{1}{\lambda} \in (0;1)$ and $w, q, z \in \triangle(X) \setminus X$. Therefore, gambling independence implies implies that $(s_j, s_{j+1}) \in R$.

The reasoning above shows that $TGI(R) \subseteq T(R)$ and from Demuynck (2009) we know that if R is transitive, $R = T(R)$. Therefore, $TGI(R) \subseteq R$, this implies that $R = TGI(R)$. □

Within-Range Independence.

Lemma A.5. $R = TWI(R)$ *if and only if* R *is transitive and satisfies within-range independence.*

Proof. ($\Rightarrow$) Consider a fixed point preference relation $R = TWI(R)$. It is transitive according to the result from Demuynck (2009). Hence, we are left to show that it satisfies within-range independence. Take $(z, z') \in R$ such that $z, z', z'' \in \triangle(I)$ for some $I \in \mathcal{I}$, then by the fact that R is a fixed point preference relation, $(\lambda z + (1-\lambda)z'', \lambda z' + (1-\lambda)z'') \in TWI(R) = R$ for every $\lambda \in [0,1]$. Take $(\lambda z + (1-\lambda)z'', \lambda z' + (1-\lambda)z'') \in R$ such that such that $z, z', z'' \in \triangle(I)$ for some $I \in \mathcal{I}$ and $\lambda \in [0,1]$, then within-range independence implies that $(z, z') \in R$. Denote by $w = \lambda z + (1-\lambda)z''$ and by $z = \lambda z' + (1-\lambda)z''$), then $z = \frac{1}{\lambda}w + \frac{1-\lambda}{\lambda}z''$ and $z' = \frac{1}{\lambda}z + \frac{1-\lambda}{\lambda}z''$ where $\frac{1}{\lambda}, \frac{1-\lambda}{\lambda} > 0$, hence, $(z, z') \in TWI(R) = R$.

($\Leftarrow$) Recall that $TWI : \mathcal{R} \to \mathcal{R}$ is monotone (every sequential closure is monotone, see Lemma A.2) and therefore, $R \subseteq TWI(R)$. Take a transitive relation R that satisfies within-range independence and let us show that $TWI(R) \subseteq R$. Recall that $(x, y) \in TWI(R)$ if there is a sequence $x = s_1, \dots, s_n = y$ such that

- $(s_j, s_{j+1}) \in R$, or
- $(\lambda s_j + (1-\lambda)z, \lambda s_{j+1} + (1-\lambda)z) \in R$ for $s_j, s_{j+1}, z \in \triangle(I)$ for some $I \in \mathcal{I}$ and some $\lambda > 0$.

Hence, $(s_j, s_{j+1}) \in R$, because preference relation satisfies within-range independence, by the same reasoning as above.

The reasoning above shows that $R \subseteq TWI(R) \subseteq T(R) \subseteq R$. This completes the proof that $R = TWI(R)$. $\square$

Proof of Corollary 2. In order to prove Corollary 2 it suffices to prove that GARP is equivalent to $>^{-1} \cap T(\geq \cup R_E) = \emptyset$ and show that $T(\geq \cup R_E)$ is separable.

Lemma A.6. *A consumption experiment satisfies GARP if and only if* $>^{-1} \cap T(\geq \cup R_E) = \emptyset$.

Proof. ($\Rightarrow$) On the contrary assume an experiment satisfies GARP and there is $(x, y) \in >^{-1} \cap T(\geq \cup R_E)$. Note that the shortest sequence that adds (x, y) to $T(\geq \cup R_E)$ is such that either s_j is a chosen point or $s_j \geq s_{j+1}$ and s_{j-1}, s_{j+1} are chosen points, where $n+1 = 1$. So, we can create the pseudo sequence $x = s'_1, \ldots, s'_m = y$, such that s'_k are chosen points for every $k \in \{1, \ldots, m-1\}$ $s'_{k+1} \in B^{\downarrow}_k$ $(y, x) \in >$ implies, that s'_m is greater than x, than $s'_1 \in \text{int}(B^{\downarrow}_{m-1})$. Note that both cases imply the violation of GARP.

($\Leftarrow$) On the contrary assume $>^{-1} \cap T(\geq \cup R_E) = \emptyset$ and it fails GARP. Then there is a sequence of chosen points, $s_1, \ldots, s_n$ such that $s_{j+1} \in B^{\downarrow}_j$ for every $j \in \{1, \ldots, n-1\}$ and $s_1 \in \text{int}(B^{\downarrow}_n)$. Note that one can decompose every pair s_j, s_{j+1} into a triple, $(s'_j, s'_{j+1}) \in R_E$ and $(s'_{j+1}, s'_{j+2}) \in \geq$. This allows us to form a pseudo sequence $s'_1, \ldots, s'_m$ such that $(s'_j, s'_{j+1}) \in (\geq \cup R_E)$. Therefore, $(x, y) \in T(\geq \cup R_E)$. Moreover, $s_1 \in \text{int}(B^{\downarrow}_n)$ implies that $(s'_m, x) \in >$ where $s'_m = s_n = y$. Hence, the $>^{-1} \cap T(\geq \cup R_E) \neq \emptyset$. $\square$

Proof of Corollary 3.

Lemma A.7. *A consumption experiment satisfies HARP if and only if* $>^{-1} \cap H(\geq \cup R_E)$.

Proof. **($\Rightarrow$)** On the contrary assume an experiment satisfies HARP and there is $(x, y) \in >^{-1} \cap H(\geq \cup R_E)$. Note that the shortest sequence that adds (x, y) to $H(\geq \cup R_E)$ is such that either $\alpha_j s_j$ is a chosen point or $\alpha_j s_j \geq \alpha_j s_{j+1}$ and $\alpha_{j-1} s_{j-1}, \alpha_{j+1} s_{j+1}$ are chosen points, where $n+1 = 1$. So, we can create the pseudo sequence $x = s'_1, \ldots, s'_m = y$, such that $\alpha_k s'_k$ are chosen points for every $k \in \{1, \ldots, m-1\}$ $\alpha_k s'_{k+1} \in B^{\downarrow}_k$. Denote by $\beta_k = \frac{\alpha_k}{\alpha_{k-1}}$ for $k \in \{2, \ldots, m-2\}$ and by $x_k = \alpha_k s_k$ for $k \in \{1, \ldots, m-2\}$, then $\frac{s_k}{\beta_k} \in B^{\downarrow}_{k+1}$ and $\prod_k \beta_k = \frac{\alpha_{m-1}}{\alpha_1}$. $(y, x) \in >$ implies, that $\alpha_{m-1}s'_m$ is greater than $x\alpha_{m-1}$ ($>$ is a homothetic relation), than $x\alpha_{m-1} = x_1 \frac{\alpha_{m-1}}{\alpha_1} = x_1 \prod_k \beta_k \in \mathrm{int}(B^{\downarrow}_{m-1})$, that is a violation of HARP.

($\Leftarrow$) On the contrary assume $>^{-1} \cap H(\geq \cup R_E) = \emptyset$ and it fails HARP. Then there is a sequence of chosen points, $s_1, \ldots, s_n$ such that $s_{j+1}/\beta_{j+1} \in B^{\downarrow}_j$ for every $j \in \{1, \ldots, n-1\}$ and $s_1 \prod_{j=2}^{n} \beta_j \in \mathrm{int}(B^{\downarrow}_n)$. Let $\alpha_1 = 1$ and $\alpha_j = \beta_j \alpha_{j-1}$, then $\alpha_n = \prod_{j=2}^{n} \beta_j$ Note that one can decompose every pair s_j, s_{j+1} into a triple, $(\alpha_j s'_j, \alpha_j s'_{j+1}) \in R_E$ and $(\alpha_j s'_{j+1}, \alpha_j s'_{j+2}) \in \geq$. Therefore, $(x, y) \in H(\geq \cup R_E)$. Moreover, $s_1 \prod_{j=2}^{n} \beta_j \in \mathrm{int}(B^{\downarrow}_n)$ implies (by homotheticity of $>$) that $(s'_m, x) \in >$ where $s'_m = s_n = y$. □

Proof of Corollary 4.

Lemma A.8. *A consumption experiment satisfies QARP if and only if* $>^{-1} \cap Q(\geq \cup R_E) = \emptyset$.

Proof. ($\Rightarrow$) On the contrary assume an experiment satisfies QARP and there is $(x, y) \in >^{-1} \cap Q(\geq \cup R_E)$. Note that the shortest sequence that adds (x, y) to $H(\geq \cup R_E)$ is such that either $\alpha_j s_j$ is a chosen point or $s_j + \alpha_j e_i \geq s_{j+1} + \alpha_j e_i$ and $s_{j-1} + \alpha_{j-1}e_i, s_{j+1} + \alpha_{j+1}e_i$ are chosen points, where $n + 1 = 1$. So, we can create the pseudo sequence $x = s'_1, \ldots, s'_m = y$, such that $s'_k + \alpha_k e_i$ are chosen points for every $k \in \{1, \ldots, m-1\}$ $s'_{k+1} + \alpha_k e_i \in B^{\downarrow}_k$. Denote by $\beta_k = \alpha_k - \alpha_{k-1}$ for $k \in \{2, \ldots, m-2\}$ and by $x_k = s_k + \alpha_k e_i$ for $k \in \{1, \ldots, m-2\}$, then $s_k - \beta_k e_i \in B^{\downarrow}_{k+1}$ and $\alpha + \sum_k \beta_k = \alpha_{m-1} - \alpha_1$; $(y, x) \in >$ implies, that $s'_m + \alpha_{m-1}e_i$ is greater than $x + \alpha_{m-1}e_i$ ($>$ is quasilinear relation), than $x + \alpha_{m-1}e_i = x_1 + \alpha_{m-1} - \alpha_1 = x_1 + \sum_k \beta_k \in \mathrm{int}(B^{\downarrow}_{m-1})$, that is a violation of QARP.

($\Leftarrow$) On the contrary assume $>^{-1} \cap Q(\geq \cup R_E) = \emptyset$ and it fails QARP. Then there is a sequence of chosen points, $s_1, \ldots, s_n$ such that $s_{j+1} -$

$\beta_{j+1}e_i \in B_j^{\downarrow}$ for every $j \in \{1,\dots,n-1\}$ and $s_1 \sum_{j=2}^{n} \beta_j \in \mathrm{int}(B_n^{\downarrow})$. Let $\alpha_1 = 0$ and $\alpha = \beta_j + \alpha_{j-1}$, then $\alpha_n = \sum_{j=2}^{n} \beta_j$ Note that one can decompose every pair s_j, s_{j+1} into a triple, $(s'_j + \alpha_j e_i, s'_{j+1} + \alpha_j e_i) \in R_E$ and $(s'_{j+1} + \alpha_j e_i, s'_{j+2} + \alpha_j e_i) \in \geq$. Therefore, $(x, y) \in Q(\geq \cup R_E)$. Moreover, $s_1 \sum_{j=2}^{n} \beta_j \in \mathrm{int}(B_n^{\downarrow})$ implies (by quasilinearity of $>$) that $(s'_m, x) \in >$ where $s'_m = s_n = y$. □

Proof of Corollary 5. We start from using the Motzkin (1951) transposition theorem. Hence, we obtain the alternative program such that either the system EU has a solution or alternative system does not have a solution.

$$\begin{cases} qA = 0, \\ q^R \geq 0, \\ q^P > 0. \end{cases}$$

Before we proceed with the proof let us state the supplementary result which would be used in the further proof.

Lemma A.9 (Demuynck and Lauwers (2009))**.** *$(x, y) \in TI(R)$ if and only if $x - y = \sum_{j=1}^{l} \beta^j (x_j - y_j)$, where $\beta^j > 0$ and $(x^j, y^j) \in R$.*

Note that Lemma A.9 already shows that external consistency is equivalent to the existence of the test we propose if A includes all possible comparisons into it, not only those which include the less preferred lotteries with support in $\mathcal{X}$.

Lemma A.10. *$P^{-1}(R) \cap TI(R) = \emptyset$ if and only if the following system has no solution*

$$\begin{cases} qA = 0, \\ q^R \geq 0, \\ q^P > 0. \end{cases}$$

Proof. ($\Rightarrow$). Assume that there is a solution to the system, and let us show that there is $(x, y) \in P^{-1}(R) \cap TI(R)$. Note that since we require $q^P > 0$ we can find $0 < q_j \in q^P$, then let $\beta_j = \frac{q}{q_j} > 0$, hence, $\beta_j = 1$. And we can rewrite the existence of the solution as $-a_j = \sum_{k \neq j} a_k$, where $a_i = x_i - y_i$. Hence, $y_j - x_j = \sum_{k\neq j} \beta_k(x_k - y_k)$, then Lemma A.9 implies that $(y, x) \in TI(R)$. At the same time a_j is a line from A^P, hence, $(x, y) \in P(R)$, this implies that $P^{-1}(R) \cap TI(R) \neq \emptyset$.

($\Leftarrow$). Assume that there is $(x, y) \in P^{-1}(R) \cap TI(R)$ and let us show that then the system has a solution. Lemma A.9 implies that there is $y - x = \sum_{j=1}^{l} \beta^j(x_j - y_j)$, hence $\sum_{j=1}^{l} \beta^j(x_j - y_j) + (x - y) = 0$, therefore, assigning $q_j = \beta_j$ and $q_k = 1$ (note that $q_k \in q^P$) for the line corresponding to $x - y$ and the rest of q to be zero we can obtain a solution if A is defined over X as the set of the individual outcomes. Therefore, to complete the proof we need to show that there is also a solution if A over $\mathcal{X}$ as the set of the individual outcomes.

To show this assume that y is such that $\text{supp}(y) \not\subseteq \mathcal{X}$, note that since $(x, y) \in R$, then $\text{supp}(x) \subseteq \mathcal{X}$, and the same is true for every x_j. Note that the only negative elements in $\sum_{k\neq j} \beta_k(x_k - y_k)$ has to be in $\mathcal{X}$, since they have to be in $\text{supp}(x) \subseteq \mathcal{X}$. Therefore, for every $k \in \{1, \ldots, l\}$, $\text{supp}(y_k) \subseteq \mathcal{X}$. At the same time since for every $k \in \{1, \ldots, l\}$, $\text{supp}(y_k) \subseteq \mathcal{X}$, $\text{supp}(x_k) \subseteq \mathcal{X}$ and $\text{supp}(x) \subseteq \mathcal{X}$, then $\text{supp}(y) \subseteq \mathcal{X}$. This completes the proof. $\square$

Proof of Corollary 6. Let us start with an analog of the Lemma A.9.

Lemma A.11. *$(x, y) \in TGI(R)$, if and only if it can be represented by $x - y = \sum \beta^k(x^k - y^k)$ with $(x^k, y^k) \in R$ and*

$$\begin{cases} \beta^k > 0 & \text{if } x^k, y^k \in \triangle(X) \setminus X, \\ \beta^k = 1 & \text{otherwise}, \end{cases}$$

such that if $x^k \in X$, then $\beta^{k+1} = 1$ and $y^{k+1} = x^k = y + \sum_{j=k}^{n} \beta^j(x^j - y^j) \in X$.

Proof. ($\Rightarrow$) If $(x, y) \in TGI(R)$, then there is a sequence $x = s_1, \ldots, s_n = y$ such that

- $(s_j, s_{j+1}) \in R$, or
- $(\lambda s_j + (1-\lambda)L, \lambda s_{j+1} + (1-\lambda)L) \in R$ for $s_j, s_{j+1}, L \in \triangle(X) \setminus X$ for some $\lambda > 0$.

Let $x_j = \lambda s_j + (1 - \lambda)L$ and $y_j = \lambda s_{j+1} + (1 - \lambda)L$. Hence, $s_j - s_{j+1} = \frac{x_j - y_j}{\lambda}$ in the second case and $s_j - s_{j+1} = x_j - x_{j+1}$ in the first case. Hence, we can represent $x - y = s_1 - s_2 + s_2 - \ldots - s_n = \sum \frac{x_j - y_j}{\lambda}$, where

$\lambda > 0$. The last set of conditions is satisfied because we are allowed to mix only lotteries. In particular $s_j = y + \sum_{j=k}^{n} \beta^j (x^j - y^j) = y^{k-1} = x^k$ by construction of $TGI(R)$.

($\Leftarrow$) Consider (x, y) such that $x - y = \sum \beta^k (x^k - y^k)$ with $(x^k, y^k) \in R$ and

$$\begin{cases} \beta^k > 0 & \text{if } x^k, y^k \in \triangle(X) \setminus X, \\ \beta^k = 1 & \text{otherwise.} \end{cases}$$

Then, $s_n = y$ and $s_j - s_{j+1} = (x_j - y_j)\beta^j$. Moreover, $x = s_1$, because $x - y = \sum \beta^k (x^k - y^k)$. Moreover, the last argument guarantees, that the mixture would be applied only to the lotteries, which can be easily checked. $\square$

The rest of proof repeats the argument from the proof of Lemma A.10.

Proof of Corollary 7. We need to state the analog of Lemma A.11 in order to complete the proof. Rest of the proof would follow the same argument as proof of Lemma A.10.

Lemma A.12. *$(x, y) \in TWI(R)$, if and only if it can be represented by $x - y = \sum \beta^k (x^k - y^k)$ with $(x^k, y^k) \in R$ and*

$$\begin{cases} \beta^k > 0 & \textit{if } x^k, y^k \in I \in \mathcal{I}, \\ \beta^k = 1 & \textit{otherwise.} \end{cases}$$

Proof. ($\Rightarrow$) If $(x, y) \in TWI(R)$, then there is a sequence $x = s_1, \ldots, s_n = y$ such that

- $(s_j, s_{j+1}) \in R$, or
- $(\lambda s_j + (1 - \lambda)L, \lambda s_{j+1} + (1 - \lambda)L) \in R$ for $s_j, s_{j+1}, L \in \triangle(I)$ for some $I \in \mathcal{I}$ and some $\lambda > 0$.

Let $x_j = \lambda s_j + (1 - \lambda)L$ and $y_j = \lambda s_{j+1} + (1 - \lambda)L$. Hence, $s_j - s_{j+1} = \frac{x_j - y_j}{\lambda}$ in the second case and $s_j - s_{j+1} = x_j - x_{j+1}$ in the first case. Hence, we can represent $x - y = s_1 - s_2 + s_2 - \ldots - s_n = \sum \frac{x_j - y_j}{\lambda}$, where $\lambda > 0$. The last set of conditions is satisfied because we are allowed to mix only lotteries.

($\Leftarrow$) Consider (x, y) such that $x - y = \sum \beta^k (x^k - y^k)$ with $(x^k, y^k) \in R$ and

$$\begin{cases} \beta^k > 0 & \text{if } x^k, y^k \in \triangle(X) \setminus X, \\ \beta^k = 1 & \text{otherwise.} \end{cases}$$

Then, $s_n = y$ and $s_j - s_{j+1} = (x_j - y_j)\beta^j$. Moreover, $x = s_1$, because $x - y = \sum \beta^k (x^k - y^k)$. □

References

Sidney N. Afriat. The construction of utility functions from expenditure data. *International Economic Review*, 8(1):67–77, 1967.

Walter Bossert. Suzumura-consistent relations: An overview. *International Journal of Economic Theory*, 14:21–34, 2018.

Walter Bossert, Yves Sprumont, and Kotaro Suzumura. Upper semi-continuous extensions of binary relations. *Journal of Mathematical Economics*, 37(3):231–246, 2002.

Marco Castillo and Mikhail Freer. A revealed preference test of quasi-linear preferences. *Working Paper*, 2016.

Christopher P. Chambers and Federico Echenique. *Revealed Preference Theory*. Cambridge University Press, 2016.

Christopher P. Chambers, Federico Echenique, and Eran Shmaya. The axiomatic structure of empirical content. *American Economic Review*, 104(8):2303–2219, 2014.

Thomas Demuynck. A general extension result with applications to convexity, homotheticity and monotonicity. *Mathematical Social Sciences*, 57(1):96–109, 2009.

Thomas Demuynck and Luc Lauwers. Nash rationalization of collective choice over lotteries. *Mathematical Social Sciences*, 57(1):1–15, 2009.

Enrico Diecidue, Ulrich Schmidt, and Peter P. Wakker. The utility of gambling reconsidered. *Journal of Risk and Uncertainty*, 29(3): 241–259, 2004.

W. Erwin Diewert. Afriat and revealed preference theory. *Review of Economic Studies*, 40(3):419–425, 1973.

John Duggan. A general extension theorem for binary relations. *Journal of Economic Theory*, 86(1):1–16, 1999.

Francoise Forges and Enrico Minelli. Afriat's theorem for general budget sets. *Journal of Economic Theory*, 144(1):135–145, 2009.

Andrei Gomberg. Revealed votes. *Social Choice and Welfare*, 51(2): 281–296, 2018.

Krzysztof Kontek and Michal Lewandowski. Range-dependent utility. *Management Science*, 2017.

Vladimir L. Levin. Measurable utility theorems for closed and lexicographic preference relations. *Soviet Mathematics, Doklady*, 27(3): 639–643, 1983.

Théodore Samuel Motzkin. Two consequences of the transposition theorem on linear inequalities. *Econometrica*, 19(2):184–185, 1951.

Hiroki Nishimura, Efe A. Ok, and John K.-H. Quah. A comprehensive approach to revealed preference theory. *American Economic Review*, 107(4):1239–1263, 2017.

Matthew Polisson, John K-H Quah, and Ludovic Renou. Revealed preferences over risk and uncertainty. Institute for Fiscal Studies Working Paper W15/25, 2015.

Marcel K. Richter. Revealed preference theory. *Econometrica*, 34(3): 635–645, 1966.

Paul A. Samuelson. A note on the pure theory of consumer's behaviour. *Economica*, 5(17):61–71, 1938.

Hal R. Varian. Non-parametric tests of consumer behaviour. *Review of Economic Studies*, 50(1):99–110, 1983.